On the Affine Schur Algebra of Type A

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Abstract

Let $n, r \in \mathbb{N}$. The affine Schur algebra $\widetilde{S}(n,r)$ (of type A) over a field K is defined to be the endomorphism algebra of certain tensor space over the extended affine Weyl group of type A_{r-1}. By the affine Schur–Weyl duality it is isomorphic to the image of the representation map of the $\mathcal{U}(\widehat{\mathfrak{gl}}_n)$ action on the tensor space when K is the field of complex numbers. We show that $\widetilde{S}(n,r)$ can be defined in another two equivalent ways. Namely, it is the image of the representation map of the semigroup algebra $K\widetilde{GL}_{n,a}$ (defined in Section 3) action on the tensor space and it equals to the 'dual' of a certain formal coalgebra related to this semigroup. By these approaches we can show many relations between different Schur algebras and affine Schur algebras and reprove one side of the affine Schur–Weyl duality.

Key words: affine Schur algebra, formal coalgebra, loop algebra, Schur algebra.

1 Introduction

The notion of the affine q-Schur algebra was first introduced by R.M.Green [11], although the algebra was first studied by V.Ginzburg and E.Vasserot [6]. For $n, r \in \mathbb{N}$, the affine q-Schur algebra $\widehat{S}_q(n,r)$ is defined as the endomorphism algebra of the q-tensor space (depending on n) over the extended affine Hecke algebra of type A_{r-1}. Specializing this approach at $q = 1$ and tensoring the resulting $\mathbb{Z}$-algebra with an infinite field K we obtain the affine Schur algebra over K (see Section 4 for detailed definition), denoted by $\widetilde{S}(n,r)$.

The purpose of this paper is to relate the affine Schur algebra to the representation theory of semigroups analogously to the finite case, and to study the relation between Schur algebras and affine Schur algebras with the same or different parameters as well as the relation between the affine complex general linear Lie algebra $\widehat{\mathfrak{gl}}_n$ and the affine complex special linear Lie algebra $\widehat{\mathfrak{sl}}_n$ and the affine Schur algebra over $\mathbb{C}$. We introduce a semigroup $\widetilde{GL}_{n,a}$ for each $a \in K^\times$ and show that it acts on an infinite dimensional tensor space $\widetilde{E}^{\otimes r}$. Denote by ϕ the corresponding representation map. We define a formal coalgebra $\widetilde{A}(n,r)$, which is a set of functions on $\widetilde{GL}_{n,a}$ (for definition of formal coalgebras see Appendix 1). The main results are the following.

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Theorem 1.1. (i) *The affine Schur algebra* $\widetilde{S}(n,r)$ *is isomorphic to the image* $\mathrm{Im}(\phi)$ *of the representation map* ϕ.

(ii) *The algebra* $\widetilde{S}(n,r)$ *is isomorphic to* $\widetilde{A}(n,r)^{\#}$ *(for a* K*-vector space* V *with a fixed basis* $\{v_i|i\in I\}$, $V^{\#}$ *is the sub* K*-vector space of* V^* *with basis dual to* $\{v_i|i\in I\}$*. See Appendix 1).*

Theorem 1.1(i)(ii) are analogous to J.A.Green's results for finite case in [7] §2.

Theorem 1.2. (i) *There is a natural algebra embedding from the Schur algebra* $S(n,r)$ *to the affine Schur algebra* $\widetilde{S}(n,r)$.

(ii) *For each* $a\in K^{\times}$ *there is a surjective algebra homomorphism* $\psi_a:\widetilde{S}(n,r)\to S(n,r)$.

(iii) *For each* $a\in K^{\times}$ *there is a surjective algebra homomorphism* $\widetilde{\det}_a^{\#}:\widetilde{S}(n,n+r)\to\widetilde{S}(n,r)$.

(iv) *These homomorphisms are compatible (for details see Theorem 5.6).*

Theorem 1.2 (i)(ii)(iv) states the relation between the Schur algebra and the affine Schur algebra. The quantized version of Theorem 1.2 (i) is [11] Proposition 2.2.5. The quantized version of Theorem 1.2 (iii) is studied by [3] [12] in finite case and by [17] [18] in affine case.

By [2] the affine Schur–Weyl duality holds. Precisely, the duality says that the quotients of $\mathcal{U}(\widehat{\mathfrak{gl}}_n)$ and the group algebra of the extended affine Weyl group of type A_{r-1} acting faithfully on the tensor space $\widetilde{E}^{\otimes r}$ centralize each other. For the quantized version we refer to [2] [5] [11] [20]. We can prove one side of the affine Schur–Weyl duality using the affine Schur algebra. Precisely, we have the following theorem.

Theorem 1.3. *Assume* $K=\mathbb{C}$, $n\geq 2$.

(i) *There is a surjective algebra homomorphism* $\widetilde{\pi}:\mathcal{U}(\widehat{\mathfrak{gl}}_n)\to\widetilde{S}(n,r)$.

(ii) *When* $r<n$, $\widetilde{\pi}|_{\mathcal{U}(\widehat{\mathfrak{sl}}_n)}:\mathcal{U}(\widehat{\mathfrak{sl}}_n)\to\widetilde{S}(n,r)$ *is surjective.*

Recently, the kernel of the quantized $\widetilde{\pi}$ is described by S.R.Doty and R.M.Green [4] in case $r<n$, and by K.McGerty [19] in general.

This paper is organized as follows. In Section 2 we briefly recall the Schur algebra defined by J.A.Green in [7] and further developed in [9] and [10] and a great many papers by others. We will generalize most of this section to the affine Schur algebra in later sections. In Section 3 we will introduce two semigroups $\widetilde{GL}_{n,a}$ and $\widetilde{SL}_{n,a}$, which are analogous to the general and special linear groups. In Section 4 we will define the affine Schur algebra in different but equivalent ways and give two multiplication formulas in terms of certain basis elements. In Sections 5–6, we will give relations between different Schur algebras and affine Schur algebras and also between the loop algebras (consequently $\widehat{\mathfrak{sl}}_n$ and $\widehat{\mathfrak{gl}}_n$) and the affine Schur algebra. As a byproduct, we obtain sets of algebra generators of the affine Schur algebra. We will define and study the formal coalgebra in Appendix 1. Appendix 2 is a generalization of J.A.Green's result (Mackey's formula) in [8] which provides necessary lemmas for the proof of the second multiplication formula in Section 4.

In the sequel K will be an infinite field. Without further comment tensor products will be taken over K.

2 The Schur algebra $S(n,r)$

This section briefly recalls on the equivalent definitions and some basic properties of the Schur algebra. The main references are [1] [7] [9].

Fix $n \in \mathbb{N}$. Let $\mathcal{M}_n = \mathcal{M}_n(K)$ be the algebra of $n \times n$ matrices with entries in K, $GL_n = GL_n(K)$ the general linear group, and $SL_n = SL_n(K)$ the special linear group.

Fix $r \in \mathbb{N}$. Let $I(n,r) = \{\underline{i} = (i_1, \ldots, i_r) | i_t \in \{1, \ldots, n\}, t = 1, \ldots, r\}$. The symmetric group Σ_r on r letters acts on $I(n,r)$ on the right by place permutation, i.e. $(i_1, \ldots, i_r)\sigma = (i_{\sigma(1)}, \ldots, i_{\sigma(r)})$ and on $I(n,r) \times I(n,r)$ diagonally. The equivalence relations on $I(n,r)$ and on $I(n,r) \times I(n,r)$ induced by this action are both denoted by the symbol $\sim_{\Sigma_r}$.

2.1 The coordinate ring

Since the field K is infinite, the coordinate functions c_{ij} on GL_n, $i,j = 1, \ldots, n$ are algebraically independent. So $A(n) = K[c_{ij}]_{i,j=1,\ldots,n}$ is a polynomial ring in n^2 indeterminates. Let $c_{\underline{i},\underline{j}}$ denote the product $c_{i_1 j_1} \cdots c_{i_r j_r}$. Then $c_{\underline{i},\underline{j}} = c_{\underline{k},\underline{l}}$ if and only if $(\underline{i},\underline{j}) \sim_{\Sigma_r} (\underline{k},\underline{l})$. The homogeneous component $A(n,r)$ of $A(n)$ of degree r has a basis $\{c_{\underline{i},\underline{j}}\}$, where $(\underline{i},\underline{j})$ runs over a set of representatives of Σ_r-orbits of $I(n,r) \times I(n,r)$. Moreover, $A(n,r)$ is a coalgebra with respect to the comultiplication $\Delta : c_{\underline{i},\underline{j}} \mapsto \sum_{\underline{s} \in I(n,r)} c_{\underline{i},\underline{s}} \otimes c_{\underline{s},\underline{j}}$ and the counit $\epsilon : A(n,r) \to K$, $c_{\underline{i},\underline{j}} \mapsto \delta_{\underline{i},\underline{j}}$. This makes $A(n)$ into a bialgebra.

The *Schur algebra*, denoted by $S(n,r)$, is defined as the dual of the coalgebra $A(n,r)$. Let $\{\xi_{\underline{i},\underline{j}}\}$ be the basis dual to $\{c_{\underline{i},\underline{j}}\}$, i.e. for $\underline{i}, \underline{j}, \underline{p}, \underline{q} \in I(n,r)$

$$\xi_{\underline{i},\underline{j}}(c_{\underline{p},\underline{q}}) = \begin{cases} 1, & \text{if} \quad (\underline{i},\underline{j}) \sim_{\Sigma_r} (\underline{p},\underline{q}) \\ 0, & \text{otherwise.} \end{cases}$$

Then $\xi_{\underline{i},\underline{j}} = \xi_{\underline{k},\underline{l}}$ if and only if $(\underline{i},\underline{j}) \sim_{\Sigma_r} (\underline{k},\underline{l})$. Multiplication is given by Schur's product rule ([7](2.3b))

$$\xi_{\underline{i},\underline{j}} \xi_{\underline{k},\underline{l}} = \sum_{(\underline{p},\underline{q}) \in (I(n,r) \times I(n,r))/\Sigma_r} Z(\underline{i},\underline{j},\underline{k},\underline{l},\underline{p},\underline{l}) \xi_{\underline{p},\underline{q}}$$

where $Z(\underline{i},\underline{j},\underline{k},\underline{l},\underline{p},\underline{l}) = \#\{\underline{s} \in I(n,r) | (\underline{i},\underline{j}) \sim_{\Sigma_r} (\underline{p},\underline{s}), \text{ and } (\underline{s},\underline{q}) \sim_{\Sigma_r} (\underline{k},\underline{l})\}$. This product rule is generalized to Coxeter system of type B by R.M.Green [13].

One observes that $\xi_{\underline{i},\underline{j}} \xi_{\underline{k},\underline{l}} = 0$ unless $\underline{j} \sim_{\Sigma_r} \underline{k}$. J.A.Green gives another version of the multiplication formula ([9](2.6))

$$\xi_{\underline{i},\underline{j}} \xi_{\underline{j},\underline{l}} = \sum_{\delta \in \Sigma_{\underline{l},\underline{j}} \backslash \Sigma_{\underline{j}} / \Sigma_{\underline{i},\underline{j}}} [\Sigma_{\underline{i},\underline{l}\delta} : \Sigma_{\underline{i},\underline{j},\underline{l}\delta}] \xi_{\underline{i},\underline{l}\delta} \tag{1}$$

where $\Sigma_{\underline{i}}$ is the stabilizer of $\underline{i}$ in Σ_r, $\Sigma_{\underline{i},\underline{l}\delta} = \Sigma_{\underline{i}} \cap \Sigma_{\underline{l}\delta}$, and $\Sigma_{\underline{i},\underline{j},\underline{l}\delta} = \Sigma_{\underline{i}} \cap \Sigma_{\underline{j}} \cap \Sigma_{\underline{l}\delta}$.

Let det$= \sum_{\sigma \in \Sigma_n} \mathrm{sgn}(\sigma) c_{(1,\ldots,n),(1,\ldots,n)\sigma} \in A(n,n)$ where sgn() is the sign function. Then multiplication by det gives an injective coalgebra homomorphism from $A(n,r)$ to $A(n,n+r)$. Taking the dual we obtain a surjective algebra homomorphism $\det^* : S(n,n+r) \to S(n,r)$.

2.2 The representation $E^{\otimes r}$ of GL_n

Let E be an n-dimensional K-vector space. Then GL_n acts on E from the left by matrix multiplication with elements in E considered as column vectors, and diagonally on the r-fold tensor product $E^{\otimes r}$ of E. The image of the corresponding representation map is isomorphic to the Schur algebra $S(n,r)$.

Therefore, there is a surjective map $e : KGL_n \to S(n,r)$. Actually, $e(g)(c) = c(g)$ for any $c \in A(n,r)$ and $g \in GL_n$. Moreover, the restriction map $e|_{KSL_n}$ is surjective and compatible with $\det^*$, i.e. $\det^* \circ e^{n+r} = e^r$ on KSL_n (see [16]).

There is an equivalence between $S(n,r)$-mod and $M(n,r)$, the category of homogeneous polynomial representations of GL_n of degree r. The space $A(n,r)$ is a homogeneous representation as well as a homogeneous anti-representation of GL_n of degree r, and therefore an $S(n,r)$-bimodule with actions ([7] (2.8b))

$$\xi \circ c = \sum \xi(c'_t)c_t, \quad c \circ \xi = \sum \xi(c_t)c'_t$$

for $\xi \in S(n,r)$, and $c \in A(n,r)$ with $\Delta(c) = \sum_t c_t \otimes c'_t$.

2.3 The representation $E^{\otimes r}$ of Σ_r

Let $\{v_1, \ldots, v_n\}$ be a K-basis of E. Then $E^{\otimes r}$ has a K-basis $\{v_{\underline{i}} = v_{i_1} \otimes \ldots \otimes v_{i_r} | \underline{i} = (i_1, \ldots, i_r) \in I(n,r)\}$. The symmetric group Σ_r acts on $E^{\otimes r}$ on the right by $v_{\underline{i}}\sigma = v_{\underline{i}\sigma}$. Then $S(n,r) \cong \mathrm{End}_{K\Sigma_r}(E^{\otimes r})$.

Moreover, if B is the quotient of $K\Sigma_r$ acting faithfully on $E^{\otimes r}$, then $\mathrm{End}_{S(n,r)}(E^{\otimes r}) \cong B$. When $n \geq r$, $B = K\Sigma_r$. This is called Schur–Weyl duality. (For details, see [15].)

2.4 The Lie algebra $\mathfrak{gl}_n$

Let $K = \mathbb{C}$ and $\mathfrak{gl}_n = \mathfrak{gl}_n(\mathbb{C})$ the complex general linear Lie algebra. Then $\mathcal{U}(\mathfrak{gl}_n)$, the universal enveloping algebra of $\mathfrak{gl}_n$, acts naturally on E, and on $E^{\otimes r}$ via the comultiplication. The image of the corresponding representation map is isomorphic to $S(n,r)$. Therefore there is a surjective homomorphism π from $\mathcal{U}(\mathfrak{gl}_n)$ to $S(n,r)$. Moreover, the restriction of this homomorphism to $\mathcal{U}(\mathfrak{sl}_n)$, the universal enveloping algebra of the complex special linear Lie algebra $\mathfrak{sl}_n$, is surjective as well and compatible with $\det^*$.

3 The semigroups $\widetilde{GL}_{n,a}$ and $\widetilde{SL}_{n,a}$

In this section we will introduce two semigroups which will be used to define the affine Schur algebra.

Fix $n \in \mathbb{N}$. Let $\mathfrak{M}_n = \mathfrak{M}_n(K) = \{M = (m_{ij})_{i,j\in\mathbb{Z}} | m_{ij} = m_{i+n,j+n} \in K$, and there are only finitely many nonzero entries in each row of $M\}$. Then with respect to matrix multiplication $\mathfrak{M}_n$ is an algebra with identity element I whose diagonal entries are 1 and off-diagonal entries are 0. Note that for $M \in \mathfrak{M}_n$ there are only finitely many nonzero entries in each column of M.

For $i,\ j \in \mathbb{Z}$, let $E_{ij} \in \mathfrak{M}_n$ be the matrix whose $(i+ln, j+ln)$ entry is 1 for all $l \in \mathbb{Z}$ and other entries are 0. Then $E_{ij} = E_{i+n,j+n}$, and $\{E_{ij} | 1 \le i \le n, j \in \mathbb{Z}\}$ is a K-basis of $\mathfrak{M}_n$. The subalgebra of $\mathfrak{M}_n$ with basis $\{E_{ij} | 1 \le i, j \le n\}$ is canonically isomorphic to $\mathcal{M}_n$. We will identify these two algebras. Moreover, $\mathfrak{M}_n \cong \mathcal{M}_n \otimes K[t, t^{-1}]$ as K-algebras, and we will identify these two algebras as well. The isomorphism is given by $E_{i,j+ln} \mapsto E_{ij} \otimes t^l$ for $i, j = 1, \ldots, n$, and $l \in \mathbb{Z}$:

$$\begin{aligned} E_{i,j+ln}E_{p,q+kn} &= \delta_{jp}E_{i,q+ln+kn} \\ (E_{ij} \otimes t^l)(E_{pq} \otimes t^k) &= \delta_{jp}E_{iq} \otimes t^{l+k}. \end{aligned}$$

For $a \in K^\times$, $s \in \mathbb{Z}$, tensoring the identity map of $\mathcal{M}_n$ with the K-algebra endomorphism $t \mapsto at^s$ of $K[t, t^{-1}]$ defines a K-algebra endomorphism $\eta_{a,s}$ of $\mathfrak{M}_n$. Precisely, $\eta_{a,s}(E_{i,j+ln}) = a^l E_{i,j+sln}$ for $i, j = 1, \ldots, n$, and $l \in \mathbb{Z}$.

Lemma 3.1. *Let $a, a' \in K^\times$, $s, s' \in \mathbb{Z}$.*

(i) *We have $\eta_{a,s} \circ \eta_{a',s'} = \eta_{a'a^{s'},ss'}$.*

(ii) *For $g \in \mathfrak{M}_n$, we have $\eta_{a,s}(g)^{tr} = \eta_{a^{-1},s}(g^{tr})$ where g^{tr} is the transpose of g.*

(iii) *The map $\eta_{a,s}$ fixes elements in $\mathcal{M}_n$.*

(iv) *If $s \neq 0$, then $\eta_{a,s}$ is injective. Moreover, $\eta_{a,\pm 1}$ are isomorphisms.*

(v) *We have $\eta_{a,0}(\mathfrak{M}_n) = \mathcal{M}_n$. We denote by η_a this map from $\mathfrak{M}_n$ to $\mathcal{M}_n$.*

Proof. (i) For $i, j = 1, \ldots, n$, $l \in \mathbb{Z}$,

$$\begin{aligned} \eta_{a,s} \circ \eta_{a',s'}(E_{i,j+ln}) &= \eta_{a,s}((a')^l E_{i,j+s'ln}) = (a')^l a^{s'l} E_{i,j+ss'ln} \\ &= \eta_{a'a^{s'}}(E_{i,j+ln}). \end{aligned}$$

(ii) For $i, j = 1, \ldots, n$, $l \in \mathbb{Z}$,

$$\begin{aligned} \eta_{a,s}(E_{i,j+ln})^{tr} &= (a^l E_{i,j+sln})^{tr} = a^l E_{j,i-sln} \\ &= \eta_{a^{-1},s}(E_{j,i-ln}) = \eta_{a^{-1},s}(E^{tr}_{i,j+ln}). \end{aligned}$$

(iii), (iv) and (v) are clear. □

For the rest of this section, we fix an $a \in K^\times$. Let $\widetilde{\det}_a = \det \circ \eta_a : \mathfrak{M}_n \to K$. The function $\widetilde{\det}_a$ is a multiplicative function.

Let $\widetilde{GL}_{n,a} = \{M \in \mathfrak{M}_n | \widetilde{\det}_a M \neq 0\}$ and $\widetilde{SL}_{n,a} = \{M \in \mathfrak{M}_n | \widetilde{\det}_a M = 1\}$. These are two semigroups containing GL_n and SL_n respectively.

Lemma 3.2. (i) *For any $b \in K^\times$, $b \neq a$, there exists $g \in \widetilde{GL}_{n,a}$ such that $\widetilde{\det}_b(g) = 0$.*

(ii) *Let $s \in \mathbb{Z}$. The restriction map $\eta_{a,s} = \eta_{a,s}|_{\widetilde{GL}_{n,a}} : \widetilde{GL}_{n,a} \to \widetilde{GL}_{n,1}$ is a semigroup homomorphism fixing elements in GL_n. Moreover, it is injective if $s \neq 0$ and bijective if $s = \pm 1$. The restriction map $\eta_a = \eta_a|_{\widetilde{GL}_{n,a}} : \widetilde{GL}_{n,a} \to GL_n$ is a surjective semigroup homomorphism fixing elements in GL_n.*

Proof. (i) Let $g = \sum_{i=1}^n (E_{ii} - bE_{i,i-n})$. Then $\widetilde{\det}_a(g) = (1 - \frac{b}{a})^n \neq 0$, but $\widetilde{\det}_b(g) = 0$.

(ii) For $g \in \widetilde{GL}_{n,a}$, it follows from Lemma 3.1 (i) that

$$\widetilde{\det}_1(\eta_{a,s}(g)) = \det \circ \eta_1 \circ \eta_{a,s}(g) = \det \circ \eta_a(g) = \widetilde{\det}_a(g) \neq 0.$$

Therefore $\eta_{a,s}(g) \in \widetilde{GL}_{n,1}$. The other statements follow directly from Lemma 3.1. □

Let G_1 be the subgroup of $\widetilde{GL}_{n,a}$ generated by $s_i = E_{i,i+1} + E_{i+1,i} + \sum_{j\neq i,i+1} E_{jj}$ ($\sum_j$ means $\sum_{j=1}^n$) where $i = 1, \dots, n-1$. Then G_1 is contained in GL_n and G_1 is isomorphic to the symmetric group Σ_n. Let G_2 be the subgroup of $\widetilde{GL}_{n,a}$ generated by $\tau_i = E_{i,i+n} + \sum_{j\neq i} E_{jj}$ where $i = 1, \dots, n$. Note that an element $\tau \in \widetilde{GL}_{n,a}$ is in G_2 if and only if τ is of the form $\tau = \sum_{s=1}^n E_{s,s+n\varepsilon_s}$ where $\varepsilon_1, \dots, \varepsilon_n \in \mathbb{Z}$. Therefore G_2 and G_1 intersect trivially since no elements in G_2 lies in GL_n except the identity matrix. Moreover $\sum_{s=1}^n E_{s,s+n\varepsilon_s} \mapsto (\varepsilon_1, \dots, \varepsilon_n)$ is an isomorphism from G_2 to $\mathbb{Z}^n$. The group G_1 acts on G_2 on the right by conjugation

$$\begin{aligned} s_i(\textstyle\sum_{s=1}^n E_{s,s+n\varepsilon_s})s_i &= E_{i,i+n\varepsilon_{i+1}} + E_{i+1,i+1+n\varepsilon_i} + \textstyle\sum_{s\neq i,i+1} E_{s,s+n\varepsilon_s} \\ &= \textstyle\sum_{s=1}^n E_{s,s+n\varepsilon_{s_i(s)}}. \end{aligned}$$

Taking the isomorphisms given above into account this right action coincides with the right action of Σ_n on $\mathbb{Z}^n$ by place permutation. Therefore we have

Lemma 3.3. *The subgroup $\langle G_1, G_2\rangle$ of $\widetilde{GL}_{n,a}$ generated by G_1 and G_2 is the semi-direct product $G_1 \ltimes G_2$ of G_1 and G_2, and hence isomorphic to $\widehat{\Sigma}_n = \Sigma_r \ltimes \mathbb{Z}^n$, the extended affine Weyl group (of type A).* □

We will identify G_1 with Σ_n, G_2 with $\mathbb{Z}^n$ and $\langle G_1, G_2\rangle$ with $\widehat{\Sigma}_n$. Consequently, the group $\widehat{\Sigma}_n$ acts on $\mathfrak{M}_n$ by conjugation. The group $\widehat{\Sigma}_n$ has another presentation (see [11]):

generators : $s_1, \dots, s_{n-1}, s_n, \rho$, where $s_n = E_{10} + E_{n,n+1} + \sum_{j\neq 1,n} E_{jj}$, $\rho = \sum_{j=1}^n E_{j,j+1}$,

relations :

$$\begin{aligned} s_i^2 &= I, \quad \text{for } i = 1, \dots, n \\ s_i s_{\overline{i+1}} s_i &= s_{\overline{i+1}} s_i s_{\overline{i+1}}, \quad \text{for } i = 1, \dots, n \\ s_j s_i &= s_i s_j, \quad \text{for } i, j = 1, \dots, n,\ j \neq \overline{i \pm 1} \\ \rho s_{\overline{i+1}} \rho^{-1} &= s_i, \quad \text{for } i = 1, \dots, n \end{aligned}$$

where $\bar{}: \mathbb{Z} \to \{1, \dots, n\}$ is the map taking least positive remainder modulo n. We claim that for $w \in \widehat{\Sigma}_n$ and $g = (g_{ij}) \in \mathfrak{M}_n$

$$wgw^{-1} = (g'_{ij} = g_{w^{-1}(i),w^{-1}(j)})$$

where for $z \in \mathbb{Z}$ and $i = 1, \dots, n$

$$\begin{aligned} \rho(z) &= z - 1, \\ s_i(z) &= \begin{cases} z & \text{if } z \not\equiv i, i+1 \pmod{n}, \\ z+1 & \text{if } z \equiv i \pmod{n}, \\ z-1 & \text{if } z \equiv i+1 \pmod{n}. \end{cases} \end{aligned}$$

The proof follows by checking on generators.

In particular,

Lemma 3.4. *The extended affine Weyl group $\widehat{\Sigma}_n$ acts on $\widetilde{GL}_{n,a}$ and $\widetilde{SL}_{n,a}$: $w \in \widehat{\Sigma}_n$ sends $g = (g_{ij})$ to $g' = wgw^{-1} = (g'_{ij})$ where $g'_{ij} = g_{w^{-1}(i),w^{-1}(j)}$.* □

Let $K^{\widetilde{GL}_{n,a}}$ be the set of maps from $\widetilde{GL}_{n,a}$ to K. It is a commutative algebra with pointwise multiplication $(ff')(g) = f(g)f'(g)$ for $f, f' \in K^{\widetilde{GL}_{n,a}}$ and $g \in \widetilde{GL}_{n,a}$. For $i, j \in \mathbb{Z}$, let c_{ij} be the coordinate function $c_{ij} : \widetilde{GL}_{n,a} \to K$, $g = (g_{pq}) \mapsto g_{ij}$. Then $c_{ij} = c_{i+n,j+n} = c_{\bar{i},j+\bar{i}-i}$.

Let $I(\mathbb{Z},r)=\{\underline{i}=(i_1,\ldots,i_r)|i_t\in\mathbb{Z},t=1,\ldots,r\}$. Then the extended affine Weyl group $\widehat{\Sigma}_r$ acts on $I(\mathbb{Z},r)$ on the right with Σ_r acting by place permutation and $\mathbb{Z}^r$ acting by shifting, i.e. $\underline{i}(\sigma,\varepsilon)=\underline{i}\sigma+n\varepsilon$ for $\underline{i}\in I(\mathbb{Z},r)$, $\sigma\in\Sigma_r$ and $\varepsilon\in\mathbb{Z}^r$. The group $\widehat{\Sigma}_r$ acts on $I(\mathbb{Z},r)\times I(\mathbb{Z},r)$ diagonally. The equivalence relations on $I(\mathbb{Z},r)$ and on $I(\mathbb{Z},r)\times I(\mathbb{Z},r)$ induced by this action will be both denoted by the symbol $\sim_{\widehat{\Sigma}_r}$. For $\underline{i}=(i_1,\ldots,i_r),\underline{j}=(j_1,\ldots,j_r)\in I(\mathbb{Z},r)$ write $c_{\underline{i},\underline{j}}=c_{i_1j_1}\cdots c_{i_rj_r}$, then $c_{\underline{i},\underline{j}}=c_{\underline{p},\underline{q}}$ if $(\underline{i},\underline{j})\sim_{\widehat{\Sigma}_r}(\underline{p},\underline{q})$.

Let $K[T]=K[t_{ij}]_{i=1,\ldots,n,j\in\mathbb{Z}}$ be the polynomial algebra in indeterminates $t_{ij},i=1,\ldots,n,j\in\mathbb{Z}$. For an integer $r\geq 0$ let $K[T]_r$ denote the homogeneous component of $K[T]$ of degree r, and let $\overline{K[T]_r}$ denote the closure of $K[T]_r$ with respect to the basis $t_{\underline{i},\underline{j}}=t_{i_1j_1}\cdots t_{i_rj_r},\underline{i}\in I(n,r),\underline{j}\in I(\mathbb{Z},r)$. We have $t_{\underline{i},\underline{j}}=t_{\underline{p},\underline{q}}$ if and only if $(\underline{i},\underline{j})\sim_{\Sigma_r}(\underline{p},\underline{q})$.

Proposition 3.5. *If P is a nonzero element in $\bigoplus_{r\geq 0}\overline{K[T]_r}$, then there exists $g\in\widetilde{GL}_{n,a}$ such that $P(c_{ij})(g)\neq 0$.* □

Proof. For $r\geq 0$ let I_r be a fixed set of representatives of $(I(n,r)\times I(n,r))/\Sigma_r$, and for $(\underline{i},\underline{j})\in I_r$ let $Z_{\underline{i},\underline{j}}$ be a fixed set of representatives of $\mathbb{Z}^r/\Sigma_{\underline{i},\underline{j}}$. Then $\{(\underline{i},\underline{j}+n\varepsilon)|(\underline{i},\underline{j})\in I_r,\varepsilon\in Z_{\underline{i},\underline{j}}\}$ is a set of representatives of $(I(n,r)\times I(\mathbb{Z},r))/\Sigma_r$. Suppose

$$P=\sum_{r\geq 0}\sum_{(\underline{i},\underline{j})\in I_r}\sum_{\varepsilon\in Z_{\underline{i},\underline{j}}}\lambda_{\underline{i},\underline{j},\varepsilon}t_{\underline{i},\underline{j}+n\varepsilon}.$$

Since P is not zero, there exists r_0, $(\underline{i}^0,\underline{j}^0)\in I_{r_0}$ and $\varepsilon^0\in Z_{\underline{i}^0,\underline{j}^0}$ such that $\lambda_{\underline{i}^0,\underline{j}^0,\varepsilon^0}\neq 0$. Set $L=\{l_1,\ldots,l_s\}=\{\varepsilon^0_1,\ldots,\varepsilon^0_{r_0}\}$, where $l_1,\ldots,l_s$ are distinct.

Let $X_{l_1},\ldots,X_{l_s}$ be a set of indeterminates labeled by the set L. Consider

$$\begin{cases}a^{l_1}X_{l_1}+\ldots+a^{l_s}X_{l_s}-1=0\\ \sum_{\varepsilon\in I(L,r_0)\cap Z_{\underline{i}^0,\underline{j}^0}}\lambda_{\underline{i}^0,\underline{j}^0,\varepsilon}X_{\varepsilon_1}\cdots X_{\varepsilon_{r_0}}\neq 0.\end{cases}\tag{2}$$

Since K is infinite, the subvariety of K^n defined by the equations (2) is nonempty (by the first equation we can express X_{l_s} as the sum of a linear combination of $X_{l_1},\ldots,X_{l_{s-1}}$ and a nonzero constant. Substituting it into the second equation we see that the resulting polynomial is not zero. Then induct on s). That is, there exist elements $x_{l_1},\ldots,x_{l_s}$ in K satisfying the equations (2). Set

$$P'=\sum_{r\geq 0}\sum_{(\underline{i},\underline{j})\in I_r}\Big(\sum_{\varepsilon\in I(L,r)\cap Z_{\underline{i},\underline{j}}}\lambda_{\underline{i},\underline{j},\varepsilon}x_{\varepsilon_1}\cdots x_{\varepsilon_r}\Big)t_{\underline{i},\underline{j}}.$$

Then P' is a nonzero polynomial in indeterminates t_{ij} $(i,j=1,\ldots,n)$. Therefore there exists $g'\in GL_n$ such that $P'(c_{ij})(g')\neq 0$.

Define $g=(g_{pq})$ to be the matrix in $\mathfrak{M}_n$ whose $(i,j+ln)$ $(i,j=1,\ldots,n,l\in\mathbb{Z})$ entry is $x_lg'_{ij}$ if $l\in L$ and 0 otherwise.

	$x_{l_1}g'$	0	$x_{l_2}g'$	0	0	$x_{l_3}g'$	
	l_1		l_2			l_3	

Therefore

$$c_{\underline{i},\underline{j}+n\varepsilon}(g)=\begin{cases}x_{\varepsilon_1}\cdots x_{\varepsilon_r}c_{\underline{i},\underline{j}}(g'), & \text{if } \varepsilon\in I(L,r),\\ 0, & \text{otherwise.}\end{cases}$$

Therefore

$$\begin{aligned}P(c_{ij})(g)&=\textstyle\sum_{r\geq 0}\sum_{(\underline{i},\underline{j})\in I_r}\sum_{\varepsilon\in Z_{\underline{i},\underline{j}}}\lambda_{\underline{i},\underline{j},\varepsilon}c_{\underline{i},\underline{j}+n\varepsilon}(g)\\ &=\textstyle\sum_{r\geq 0}\sum_{(\underline{i},\underline{j})\in I_r}\sum_{\varepsilon\in I(L,r)\cap Z_{\underline{i},\underline{j}}}\lambda_{\underline{i},\underline{j},\varepsilon}c_{\underline{i},\underline{j}+n\varepsilon}(g)\\ &=\textstyle\sum_{r\geq 0}\sum_{(\underline{i},\underline{j})\in I_r}\big(\sum_{\varepsilon\in I(L,r)\cap Z_{\underline{i},\underline{j}}}\lambda_{\underline{i},\underline{j},\varepsilon}x_{\varepsilon_1}\cdots x_{\varepsilon_r}\big)c_{\underline{i},\underline{j}}(g)\\ &=P'(c_{ij})(g')\neq 0.\end{aligned}$$

By definition we have $g=\sum_{i,j=1,\ldots,n,l\in L}x_l g'_{ij}E_{i,j+ln}$. Therefore

$$\eta_a(g)=\sum_{i,j=1,\ldots,n}(\sum_{l\in L}a^l x_l)g'_{ij}E_{ij}=g',$$

which implies $g\in\widetilde{GL}_{n,a}$. $\square$

If P in Proposition 3.5 is homogeneous, i.e. $P\in\overline{K[T]_r}$ for some r, then the polynomial P' in the proof is homogeneous of degree r. In this case, we can choose g' to be in SL_n, and then the resulting g lies in $\widetilde{SL}_{n,a}$. Namely, we have the following proposition.

Proposition 3.6. *Let $r\geq 0$ be an integer. If P is a nonzero element in $\overline{K[T]_r}$, then there exists $g\in\widetilde{SL}_{n,a}$ such that $P(c_{ij})(g)\neq 0$.* $\square$

By Proposition 3.5 the functions $\{c_{ij}|i=1,\ldots,n,j\in\mathbb{Z}\}$ are algebraically independent, and hence the subalgebra $\widetilde{A}(n)$ of $K^{\widetilde{GL}_{n,a}}$ generated by all c_{ij}'s is a polynomial algebra in indeterminates $\{c_{ij}\}$, where (i,j) runs over a set of representatives of $\widehat{\Sigma}_1$-orbits of $\mathbb{Z}\times\mathbb{Z}$ (here $\widehat{\Sigma}_1=\mathbb{Z}$ acts on $\mathbb{Z}=I(\mathbb{Z},1)$ by shifting and on $\mathbb{Z}\times\mathbb{Z}$ diagonally), for example we may take $\{(i,j)|i=1,\ldots,n,j\in\mathbb{Z}\}$. Let $r\geq 0$ be an integer. Denote by $\widetilde{A}(n,r)$ the homogeneous component of $\widetilde{A}(n)$ of degree r. Note that $c_{\underline{i},\underline{j}}=c_{\underline{p},\underline{q}}$ if and only if $(\underline{i},\underline{j})\sim_{\widehat{\Sigma}_r}(\underline{p},\underline{q})$. Thus $\widetilde{A}(n,r)$ has a basis $\{c_{\underline{i},\underline{j}}\}$, where $(\underline{i},\underline{j})$ runs over a set of representatives of $\widehat{\Sigma}_r$-orbits of $I(\mathbb{Z},r)\times I(\mathbb{Z},r)$. By Proposition 3.5 the space $\bigoplus_{r\geq 0}\overline{\widetilde{A}(n,r)}$ is a subspace of $K^{\widetilde{GL}_{n,a}}$.

4 The affine Schur algebra $\widetilde{S}(n,r)$

Fix n, $r\in\mathbb{N}$. In this section we will define the affine Schur algebra and show how it is related to the two semigroups $\widetilde{GL}_{n,a}$ and $\widetilde{SL}_{n,a}$ and the extended affine Weyl group $\widehat{\Sigma}_r$.

Let, for a moment, G be any semigroup with identity 1_G. Let K^G be the K-space of all maps from G to K. Then K^G is a commutative K-algebra. The semigroup structure on G gives rise to two maps

$$\Delta=\Delta_G:K^G\to K^{G\times G},\quad f\mapsto((s,t)\mapsto f(st)\text{ for }s,t\in G)$$
$$\epsilon=\epsilon_G:K^G\to K,\quad f\mapsto f(1_G).$$

Both Δ and ϵ are K-algebra homomorphisms (see [7]).

The restrictions of Δ and ϵ to $\widetilde{A}(n,r)$ can be written explicitly as follows

$$\Delta: \widetilde{A}(n,r) \to \overline{\widetilde{A}(n,r) \otimes \widetilde{A}(n,r)}, \quad c_{\underline{i},\underline{j}} \mapsto \textstyle\sum_{\underline{s}\in I(\mathbb{Z},r)} c_{\underline{i},\underline{s}} \otimes c_{\underline{s},\underline{j}}$$
$$\epsilon: \widetilde{A}(n,r) \to K, \quad c_{\underline{i},\underline{j}} \mapsto \delta_{\underline{i},\underline{j}}.$$

Lemma 4.1. *These two maps make $\widetilde{A}(n,r)$ into a formal K-coalgebra (for definition of formal coalgebras see Appendix 1).*

Proof. One can check by the definition of formal coalgebras. □

It follows by Theorem 7.6 that $\widetilde{S}(n,r) = \widetilde{A}(n,r)^{\#}$ is a K-algebra, called *the affine Schur algebra*. Let $\{\xi_{\underline{i},\underline{j}} | \underline{i},\underline{j} \in I(\mathbb{Z},r)\}$ be the basis dual to $\{c_{\underline{i},\underline{j}} | \underline{i},\underline{j} \in I(\mathbb{Z},r)\}$, i.e.

$$\xi_{\underline{i},\underline{j}}(c_{\underline{p},\underline{q}}) = \begin{cases} 1, & \text{if } \ (\underline{i},\underline{j}) \sim_{\widehat{\Sigma}_r} (\underline{p},\underline{q}) \\ 0, & \text{otherwise.} \end{cases}$$

Therefore $\xi_{\underline{i},\underline{j}} = \xi_{\underline{p},\underline{q}}$ if and only if $(\underline{i},\underline{j}) \sim_{\widehat{\Sigma}_r} (\underline{p},\underline{q})$.

Recall that $^{-}\colon \mathbb{Z} \to \{1,\dots,n\}$ is the map taking least positive remainder modulo n. It can be extended to $^{-}\colon I(\mathbb{Z},r) \to I(n,r)$. Then $\xi_{\underline{i},\underline{j}} = \xi_{\overline{\underline{i}},\underline{j}+\overline{\underline{i}}-\underline{i}} = \xi_{\underline{i}+\overline{\underline{j}}-\underline{j},\overline{\underline{j}}}$. In some later cases we will assume that $\underline{i} \in I(n,r)$ or $\underline{j} \in I(n,r)$.

For $\underline{i},\underline{j},\underline{k},\underline{l} \in I(\mathbb{Z},r)$, we have the following formula, known as Schur's product rule:

$$\xi_{\underline{i},\underline{j}}\xi_{\underline{k},\underline{l}} = \sum_{(\underline{p},\underline{q})\in(I(\mathbb{Z},r)\times I(\mathbb{Z},r))/\widehat{\Sigma}_r} Z(\underline{i},\underline{j},\underline{k},\underline{l},\underline{p},\underline{q})\xi_{\underline{p},\underline{q}}$$

where $Z(\underline{i},\underline{j},\underline{k},\underline{l},\underline{p},\underline{q}) = \#\{\underline{s} \in I(\mathbb{Z},r) | (\underline{i},\underline{j}) \sim_{\widehat{\Sigma}_r} (\underline{p},\underline{s}), (\underline{s},\underline{q}) \sim_{\widehat{\Sigma}_r} (\underline{k},\underline{l})\}$.

Directly from Schur's product rule, we have the following proposition.

Proposition 4.2. (i) *We have $\xi_{\underline{i},\underline{j}}\xi_{\underline{k},\underline{l}} = 0$ unless $\underline{j} \sim_{\widehat{\Sigma}_r} \underline{k}$.*

(ii) *We have $\xi_{\underline{i},\underline{i}}\xi_{\underline{i},\underline{j}} = \xi_{\underline{i},\underline{j}} = \xi_{\underline{i},\underline{j}}\xi_{\underline{j},\underline{j}}$, for $\underline{i},\underline{j} \in I(\mathbb{Z},r)$.*

(iii) *$\sum_{\underline{i}\in I(n,r)/\Sigma_r} \xi_{\underline{i},\underline{i}}$ is a decomposition of unity into orthogonal idempotents.*

(iv) *The subalgebra of $\widetilde{S}(n,r)$ with basis $\{\xi_{\underline{i},\underline{j}} | \underline{i},\underline{j} \in I(n,r)\}$ is naturally isomorphic to $S(n,r)$. We will identify these two algebras.*

Let $\widetilde{e}_a : K\widetilde{GL}_{n,a} \to \widetilde{S}(n,r)$ be the algebra homomorphism sending $g \in \widetilde{GL}_{n,a}$ to $(\widetilde{e}_a(g) : c \mapsto c(g))$ for any $c \in \widetilde{A}(n,r)$. The image of KGL_n under $\widetilde{e}_a$ lies in $S(n,r)$. In fact, the restriction map $\widetilde{e}_a|_{KGL_n}$ is the map e defined in Section 2.2. The following theorem follows from Proposition 3.6.

Theorem 4.3. *The map $\widetilde{e}_a$ is surjective. Moreover, the restriction map $\widetilde{e}_a|_{K\widetilde{SL}_{n,a}} : K\widetilde{SL}_{n,a} \to \widetilde{S}(n,r)$ is surjective.*

Proof. Suppose the image of the restriction map $\widetilde{e}_a|_{K\widetilde{SL}_{n,a}}$ is a proper subspace of $\widetilde{S}(n,r)$. Then there exists a nonzero $c \in \overline{\widetilde{A}(n,r)} = S(n,r)^*$ such that $c(\xi) = 0$ for any $\xi \in \operatorname{Im} \widetilde{e}_a|_{K\widetilde{SL}_{n,a}}$. In particular, $c(g) = c(\widetilde{e}_a(g)) = 0$ for any $g \in \widetilde{SL}_{n,a}$, contradicting Proposition 3.6. □

Let $\widetilde{Y}_a$ be the kernel of $\widetilde{e}_a$, i.e. $0 \to \widetilde{Y}_a \to K\widetilde{GL}_{n,a} \xrightarrow{\widetilde{e}_a} \widetilde{S}(n,r) \to 0$ is exact.

Proposition 4.4. *Let* $f \in K^{\widetilde{GL}_{n,a}}$. *Then* $f \in \overline{\widetilde{A}(n,r)}$ *if and only if* $f(\widetilde{Y}_a) = 0$.

Proof. Assume $f \in \overline{\widetilde{A}(n,r)}$. For any $y \in \widetilde{Y}_a$, we have $\widetilde{e}_a(y) = 0$, i.e. $\widetilde{e}_a(y)(c_{\underline{i},\underline{j}}) = c_{\underline{i},\underline{j}}(y) = 0$ for any $\underline{i} \in I(n,r)$, $\underline{j} \in I(\mathbb{Z},r)$. Hence $f(y) = 0$.

Assume $f(\widetilde{Y}_a) = 0$. Then there exists $c \in \overline{\widetilde{A}(n,r)} = S(n,r)^*$ such that $c(g) = \widetilde{e}_a(g)(c) = c(\widetilde{e}_a(g)) = f(g)$ for any $g \in \widetilde{GL}_{n,a}$, and hence $f = c$. □

Let W be a representation of $\widetilde{GL}_{n,a}$ with basis $\{w_j | j \in J\}$. Suppose $g(w_j) = \sum_{j' \in J} r_{j'j}(g) w_{j'}$, for $g \in \widetilde{GL}_{n,a}$, $j \in J$. We say that W is a representation with coefficients in $\overline{\widetilde{A}(n,r)}$ if $r_{j'j} \in \overline{\widetilde{A}(n,r)}$ for any $j, j' \in J$. The matrix $(r_{j'j})_{j,j' \in J}$ is called the coefficient matrix of W. Let $\widetilde{M}(n,r)$ denote the category of representations of $\widetilde{GL}_{n,a}$ with coefficients in $\overline{\widetilde{A}(n,r)}$.

Proposition 4.5. *Let* W *be a representation of* $\widetilde{GL}_{n,a}$. *Then* $W \in \widetilde{M}(n,r)$ *if and only if* $\widetilde{Y}_a W = 0$. *This induces an equivalence between* $\widetilde{M}(n,r)$ *and* $\widetilde{S}(n,r)$-Mod.

Proof. Let $\{w_j | j \in J\}$ be the basis of W, and $(r_{j'j})_{j,j' \in J}$ the coefficient matrix of W with respect to this basis. Then $W \in \widetilde{M}(n,r)$ if and only if $r_{j'j} \in \overline{\widetilde{A}(n,r)}$. By Proposition 4.4, this is equivalent to $\widetilde{Y}_a W = 0$. □

Let $\widetilde{E} = K\{v_i | i \in \mathbb{Z}\}$. Then $\widetilde{E}^{\otimes r} = K\{v_{\underline{i}} = v_{i_1} \otimes \ldots \otimes v_{i_r} | \underline{i} = (i_1, \ldots, i_r) \in I(\mathbb{Z},r)\}$. The semigroup $\widetilde{GL}_{n,a}$ acts on $\widetilde{E}$ from the left by matrix multiplication, and on $\widetilde{E}^{\otimes r}$ diagonally, i.e. $g(v_{\underline{i}}) = \sum_{\underline{j} \in I(\mathbb{Z},r)} c_{\underline{j},\underline{i}}(g) v_{\underline{j}}$ for $\underline{i} \in I(\mathbb{Z},r)$. By Proposition 4.5, the tensor space $\widetilde{E}^{\otimes r}$ can be regarded as an $\widetilde{S}(n,r)$-module, $\xi(v_{\underline{i}}) = \sum_{\underline{j} \in I(\mathbb{Z},r)} \xi(c_{\underline{j},\underline{i}}) v_{\underline{j}}$. This is a faithful module. Thus the representation map $\phi : \widetilde{S}(n,r) \to \mathrm{End}_K(\widetilde{E}^{\otimes r})$ is injective.

The extended affine Weyl group $\widehat{\Sigma}_r$ acts on $\widetilde{E}^{\otimes r}$ on the right by $v_{\underline{i}} w = v_{\underline{i}w}$. These two actions commute. In particular, the image of ϕ lies in $\mathrm{End}_{K\widehat{\Sigma}_r}(\widetilde{E}^{\otimes r})$. In fact we have

Theorem 4.6. *The image of* ϕ *is exactly* $\mathrm{End}_{\mathrm{K}\widehat{\Sigma}_\mathrm{r}}(\widetilde{\mathrm{E}}^{\otimes \mathrm{r}})$. *Therefore* ϕ *induces an isomorphism between* $\widetilde{S}(n,r)$ *and* $\mathrm{End}_{\mathrm{K}\widehat{\Sigma}_\mathrm{r}}(\widetilde{\mathrm{E}}^{\otimes \mathrm{r}})$.

Proof. For $\underline{i}$, $\underline{j} \in I(\mathbb{Z},r)$, let $x_{\underline{i},\underline{j}}$ denote the endomorphism of $\widetilde{E}^{\otimes r}$ sending $v_{\underline{l}}$ to $\delta_{\underline{j},\underline{l}} v_{\underline{i}}$. Then

$$\mathrm{End}_K(\widetilde{E}^{\otimes r}) = \{ \sum_{\underline{i},\underline{j} \in I(\mathbb{Z},r)} m_{\underline{i},\underline{j}} x_{\underline{i},\underline{j}} | m_{\underline{i},\underline{j}} \in K \text{ and for each } \underline{j}, m_{\underline{i},\underline{j}} = 0 \text{ for almost all } \underline{i} \}$$

Since

$$\xi_{\underline{i},\underline{j}} v_{\underline{l}} = \sum_{\underline{k} \in I(\mathbb{Z},r)} \xi_{\underline{i},\underline{j}}(c_{\underline{k},\underline{l}}) v_{\underline{k}} = \sum_{w \in \widehat{\Sigma}_{\underline{i},\underline{j}} \backslash \widehat{\Sigma}_r : \underline{j}w = \underline{l}} v_{\underline{i}w},$$

the image of $\xi_{\underline{i},\underline{j}}$ under ϕ is

$$\phi(\xi_{\underline{i},\underline{j}}) = \sum_{w \in \widehat{\Sigma}_{\underline{i},\underline{j}} \backslash \widehat{\Sigma}_r} x_{\underline{i}w,\underline{j}w}.$$

The right action of $\widehat{\Sigma}_r$ on $\widetilde{E}^{\otimes r}$ induces a right action of $\widehat{\Sigma}_r$ on $\mathrm{End}_K(\widetilde{E}^{\otimes r})$,

$$f^w = \sum_{\underline{i},\underline{j} \in I(\mathbb{Z},r)} m_{\underline{i},\underline{j}} x_{\underline{i}w,\underline{j}w} = \sum_{\underline{i},\underline{j} \in I(\mathbb{Z},r)} m_{\underline{i}w^{-1},\underline{j}w^{-1}} x_{\underline{i},\underline{j}}$$

for $f=\sum_{\underline{i},\underline{j}\in I(\mathbb{Z},r)} m_{\underline{i},\underline{j}}x_{\underline{i},\underline{j}} \in \mathrm{End}_K(\widetilde{E}^{\otimes r})$ and $w\in\widehat{\Sigma}_r$. It is easy to see f is fixed by $\widehat{\Sigma}_r$ if and only if $m_{\underline{i},\underline{j}}=m_{\underline{i}w,\underline{j}w}$ for any $\underline{i},\underline{j}\in I(\mathbb{Z},r)$ and $w\in\widehat{\Sigma}_r$, that is, f is a linear combination of $\phi(\xi_{\underline{i},\underline{j}})$'s. Now the desired result follows from $\mathrm{End}_{K\widehat{\Sigma}_r}(\widetilde{E}^{\otimes r})=(\mathrm{End}_K(\widetilde{E}^{\otimes r}))^{\widehat{\Sigma}_r}$. $\square$

We will identify $\phi(\xi_{\underline{i},\underline{j}})$ with $\xi_{\underline{i},\underline{j}}$. Now let us give an analogue of J.A.Green's product formula (1)([9](2.6)) for the finite Schur algebra.

For $\underline{i}$, $\underline{j}\in I(\mathbb{Z},r)$, let $\widehat{\Sigma}_{\underline{i}}$ be the stabilizer of $\underline{i}$ in $\widehat{\Sigma}_r$ and $\widehat{\Sigma}_{\underline{i},\underline{j}}=\widehat{\Sigma}_{\underline{i}}\cap\widehat{\Sigma}_{\underline{j}}$, and so on. Note that if $\underline{i}\in I(n,r)$, then $\widehat{\Sigma}_{\underline{i}}=\Sigma_{\underline{i}}$.

Let $A=K\{x_{\underline{i},\underline{j}}|\underline{i},\underline{j}\in I(\mathbb{Z},r)\}$, $B=\mathrm{End}_K(\widetilde{E}^{\otimes r})$. For a subgroup H of $\widehat{\Sigma}_r$, let

$$\begin{aligned} A_H &= \{a\in A|a^h=a \text{ for any } h\in H\}\\ B_H &= \{b\in B|b^h=b \text{ for any } h\in H\}\end{aligned}$$

then

$$T_{H,\widehat{\Sigma}_r}: A_H\to B_H,\quad a\mapsto \sum_{g\in H\backslash\widehat{\Sigma}_r} a^g$$

is well-defined (see Appendix 2). For $\delta\in\widehat{\Sigma}_r$ we set $H^\delta=\delta^{-1}H\delta$.

The following lemma is a special case of Appendix 2 Lemma 8.4: Mackey's formula.

Lemma 4.7. *Let H_1 and H_2 be two subgroups of $\widehat{\Sigma}_r$, and $a\in A_{H_1}$, $b\in A_{H_2}$. If $aT_{H_2,\widehat{\Sigma}_r}(b)\in A_{H_1}$, $T_{H_1\cap H_2^\delta,H_1}(ab^\delta)=0$ for almost all δ, and $T_{H_1\cap H_2^\delta,H_1}(ab^\delta)\in A_{H_1}$ for all δ, then*

$$T_{H_1,\widehat{\Sigma}_r}(a)T_{H_2,\widehat{\Sigma}_r}(b)=\sum_{\delta\in H_2\backslash\widehat{\Sigma}_r/H_1} T_{H_1\cap H_2^\delta,\widehat{\Sigma}_r}(ab^\delta).$$

$\square$

Let $\underline{i},\underline{j},\underline{l}\in I(\mathbb{Z},r)$. Then

$$\begin{aligned}\xi_{\underline{i},\underline{j}} &= T_{\widehat{\Sigma}_{\underline{i},\underline{j}},\widehat{\Sigma}_r}(x_{\underline{i},\underline{j}})\\ \xi_{\underline{j},\underline{l}} &= T_{\widehat{\Sigma}_{\underline{j},\underline{l}},\widehat{\Sigma}_r}(x_{\underline{j},\underline{l}}).\end{aligned}$$

Therefore setting $H_1=\widehat{\Sigma}_{\underline{i},\underline{j}}$, $H_2=\widehat{\Sigma}_{\underline{j},\underline{l}}$, $a=x_{\underline{i},\underline{j}}$, $b=x_{\underline{j},\underline{l}}$ and applying Lemma 4.7, we obtain

$$\begin{aligned}\xi_{\underline{i},\underline{j}}\xi_{\underline{j},\underline{l}} &= T_{\widehat{\Sigma}_{\underline{i},\underline{j}},\widehat{\Sigma}_r}(x_{\underline{i},\underline{j}})T_{\widehat{\Sigma}_{\underline{j},\underline{l}},\widehat{\Sigma}_r}(x_{\underline{j},\underline{l}})\\
&= \textstyle\sum_{\delta\in\widehat{\Sigma}_{\underline{j},\underline{l}}\backslash\widehat{\Sigma}_r/\widehat{\Sigma}_{\underline{i},\underline{j}}} T_{\widehat{\Sigma}_{\underline{i},\underline{j}}\cap\widehat{\Sigma}^\delta_{\underline{j},\underline{l}},\widehat{\Sigma}_r}(x_{\underline{i},\underline{j}}x^\delta_{\underline{j},\underline{l}})\\
&= \textstyle\sum_{\delta\in\widehat{\Sigma}_{\underline{j},\underline{l}}\backslash\widehat{\Sigma}_r/\widehat{\Sigma}_{\underline{i},\underline{j}}} T_{\widehat{\Sigma}_{\underline{i},\underline{j}}\cap\widehat{\Sigma}_{\underline{j}\delta,\underline{l}\delta},\widehat{\Sigma}_r}(x_{\underline{i},\underline{j}}x_{\underline{j}\delta,\underline{l}\delta})\\
&= \textstyle\sum_{\delta\in\widehat{\Sigma}_{\underline{j},\underline{l}}\backslash\widehat{\Sigma}_{\underline{j}}/\widehat{\Sigma}_{\underline{i},\underline{j}}} T_{\widehat{\Sigma}_{\underline{i},\underline{j},\underline{l}\delta},\widehat{\Sigma}_r}(x_{\underline{i},\underline{l}\delta})\\
&= \textstyle\sum_{\delta\in\widehat{\Sigma}_{\underline{j},\underline{l}}\backslash\widehat{\Sigma}_{\underline{j}}/\widehat{\Sigma}_{\underline{i},\underline{j}}} T_{\widehat{\Sigma}_{\underline{i},\underline{l}\delta},\widehat{\Sigma}_r}T_{\widehat{\Sigma}_{\underline{i},\underline{j},\underline{l}\delta},\widehat{\Sigma}_{\underline{i},\underline{l}\delta}}(x_{\underline{i},\underline{l}\delta})\\
&= \textstyle\sum_{\delta\in\widehat{\Sigma}_{\underline{j},\underline{l}}\backslash\widehat{\Sigma}_{\underline{j}}/\widehat{\Sigma}_{\underline{i},\underline{j}}}[\widehat{\Sigma}_{\underline{i},\underline{l}\delta}:\widehat{\Sigma}_{\underline{i},\underline{j},\underline{l}\delta}]\xi_{\underline{i},\underline{l}\delta}\end{aligned}$$

where the second to last equality follows by Lemma 8.2.

The following version only involves the symmetric group Σ_r, and hence easier to calculate.

Corollary 4.8. *For $\underline{i},\underline{j},\underline{l}\in I(n,r)$, $\varepsilon,\varepsilon'\in\mathbb{Z}^r$, we have*

$$\xi_{\underline{i},\underline{j}+n\varepsilon}\xi_{\underline{j},\underline{l}+n\varepsilon'}=\sum_{\delta\in\Sigma_{\underline{j},\underline{l},\varepsilon'}\backslash\Sigma_{\underline{j}}/\Sigma_{\underline{i},\underline{j},\varepsilon}}[\Sigma_{\underline{i},\underline{l}\delta,\varepsilon'\delta+\varepsilon}:\Sigma_{\underline{i},\underline{j},\underline{l}\delta,\varepsilon'\delta,\varepsilon}]\xi_{\underline{i},\underline{l}\delta+n(\varepsilon'\delta+\varepsilon)}.$$

Proof. We have

$$
\begin{aligned}
\xi_{\underline{i},\underline{j}+n\varepsilon}\xi_{\underline{j},\underline{l}+n\varepsilon'} &= \xi_{\underline{i}-n\varepsilon,\underline{j}}\xi_{\underline{j},\underline{l}+n\varepsilon'} \\
&= \sum_{\delta\in\widehat{\Sigma}_{\underline{j},\underline{l}+n\varepsilon'}\backslash\widehat{\Sigma}_{\underline{j}}/\widehat{\Sigma}_{\underline{i}-n\varepsilon,\underline{j}}}[\widehat{\Sigma}_{\underline{i}-n\varepsilon,(\underline{l}+n\varepsilon')\delta} : \widehat{\Sigma}_{\underline{i}-n\varepsilon,\underline{j},(\underline{l}+n\varepsilon')\delta}]\xi_{\underline{i}-n\varepsilon,(\underline{l}+n\varepsilon')\delta} \\
&= \sum_{\delta\in\Sigma_{\underline{j},\underline{l},\varepsilon'}\backslash\Sigma_{\underline{j}}/\Sigma_{\underline{i},\underline{j},\varepsilon}}[\widehat{\Sigma}^{\varepsilon}_{\underline{i}-n\varepsilon,\underline{l}\delta+n\varepsilon'\delta} : \widehat{\Sigma}^{\varepsilon}_{\underline{i}-n\varepsilon,\underline{j},\underline{l}\delta+n\varepsilon'\delta}]\xi_{\underline{i},\underline{l}\delta+n(\varepsilon'\delta+\varepsilon)} \\
&= \sum_{\delta\in\Sigma_{\underline{j},\underline{l},\varepsilon'}\backslash\Sigma_{\underline{j}}/\Sigma_{\underline{i},\underline{j},\varepsilon}}[\widehat{\Sigma}_{\underline{i},\underline{l}\delta+n(\varepsilon'\delta+\varepsilon)} : \widehat{\Sigma}_{\underline{i},\underline{j}+n\varepsilon,\underline{l}\delta+n(\varepsilon'\delta+\varepsilon)}]\xi_{\underline{i},\underline{l}\delta+n(\varepsilon'\delta+\varepsilon)} \\
&= \sum_{\delta\in\Sigma_{\underline{j},\underline{l},\varepsilon'}\backslash\Sigma_{\underline{j}}/\Sigma_{\underline{i},\underline{j},\varepsilon}}[\Sigma_{\underline{i},\underline{l}\delta,\varepsilon'\delta+\varepsilon} : \Sigma_{\underline{i},\underline{j},\varepsilon,\underline{l}\delta,\varepsilon'\delta+\varepsilon}]\xi_{\underline{i},\underline{l}\delta+n(\varepsilon'\delta+\varepsilon)} \\
&= \sum_{\delta\in\Sigma_{\underline{j},\underline{l},\varepsilon'}\backslash\Sigma_{\underline{j}}/\Sigma_{\underline{i},\underline{j},\varepsilon}}[\Sigma_{\underline{i},\underline{l}\delta,\varepsilon'\delta+\varepsilon} : \Sigma_{\underline{i},\underline{j},\underline{l}\delta,\varepsilon'\delta,\varepsilon}]\xi_{\underline{i},\underline{l}\delta+n(\varepsilon'\delta+\varepsilon)}
\end{aligned}
$$

as desired. □

5 Maps between affine Schur algebras

In this section we will study relations between Schur algebras and affine Schur algebras.

Let G be a semigroup with identity 1_G. Recall that K^G is an algebra and $\Delta_G : K^G \to K^{G\times G}$ and $\epsilon_G : K^G \to K$ are algebra homomorphisms. Let A be a subspace of K^G with a fixed basis $\{a_i\}_{i\in I}$ such that for a fixed $g \in G$ almost all $a_i(g) = 0$ $(i \in I)$. Assume that $\overline{A}$ is a subspace of K^G via $(\sum_{i\in I}\lambda_i a_i)(g) = \sum_{i\in I}\lambda_i a_i(g)$. Then $\overline{A\otimes A}$ is considered as a subspace of $K^{G\times G}$ via $(\sum_{i,j\in I}\lambda_{ij}a_i\otimes a_j)(g,g') = \sum_{i,j\in I}\lambda_{ij}a_i(g)a_j(g')$. For a proof, note that for $g \in G$ the function $\sum_{i,j\in I}\lambda_{ij}a_i(g)a_j$ lies in K^G. If $\sum_{i,j\in I}\lambda_{ij}a_i\otimes a_j$ considered as a function on $G\times G$ is 0, then the coefficient $\sum_{i\in I}\lambda_{ij}a_i(g)$ of a_j equals to 0, and hence $\lambda_{ij} = 0$.

We call A a sub formal coalgebra of K^G if in addition A is a formal coalgebra with respect to $\Delta_G|_A$ and $\epsilon_G|_A$. Let A be a sub formal coalgebra of K^G. Define the evaluation map $e_G : KG \to A^{\#}$, $g \mapsto (e_G(g) : a \mapsto a(g), a \in A)$. This is a surjective algebra homomorphism. Suppose on the contrary that $\mathrm{Im}(e_G)$ is a proper subspace of $A^{\#}$. Then there exists a nonzero element c in $\overline{A} = (A^{\#})^*$ such that $c(\mathrm{Im}(e_G)) = 0$. In particular, $c(e_G(g)) = 0$ for any $g \in G$, i.e. $c(g) = 0$ for any $g \in G$. Thus $c = 0$ in K^G, contradicting the assumption that $\overline{A}$ is a subspace of K^G.

Let $\phi : G \to H$ be a homomorphism of semigroups. Denote by ϕ^* the algebra homomorphism from K^H to K^G sending f to $f\circ\phi$. If ϕ is injective then ϕ^* is surjective; if ϕ is surjective then ϕ^* is injective. Moreover, if $\psi : H \to L$ is also a semigroup homomorphism, then $(\psi\circ\phi)^* = \phi^*\circ\psi^*$.

For $f \in K^H$ and $g, g' \in G$, we have

$$
\begin{aligned}
(\Delta_G\circ\phi^*(f))(g,g') &= \phi^*(f)(gg') = f(\phi(gg')) = f(\phi(g)\phi(g')) \\
&= \Delta_H(f)(\phi(g),\phi(g')) = ((\phi\times\phi)^*\circ\Delta_H(f))(g,g') \\
\epsilon_G\circ\phi^*(f) &= \phi^*(f)(1_G) = f(\phi(1_G)) = f(1_H) = \epsilon_H(f).
\end{aligned}
$$

Hence $\Delta_G\circ\phi^* = (\phi\times\phi)^*\circ\Delta_H$, and $\epsilon_G\circ\phi^* = \epsilon_H$. That is, the following diagrams commute:

$$
\begin{array}{ccc}
K^H & \xrightarrow{\phi^*} & K^G \\
\downarrow{\scriptstyle \Delta_H} & & \downarrow{\scriptstyle \Delta_G} \\
K^{H\times H} & \xrightarrow{(\phi\times\phi)^*} & K^{G\times G}
\end{array}
\qquad\qquad
\begin{array}{ccc}
K^H & \xrightarrow{\phi^*} & K^G \\
\downarrow{\scriptstyle \epsilon_H} & & \downarrow{\scriptstyle \epsilon_G} \\
K & \xrightarrow{\mathrm{id}} & K
\end{array}
$$

Let A and B be a sub formal coalgebras of K^G and K^H respectively. Suppose $\phi^*(B) \subseteq \overline{A}$ and $\phi^*|_B : B \to \overline{A}$ is row finite. Then $\phi^*|_B$ is a formal homomorphism of formal coalgebras (see Definition 7.5) and $\overline{\phi^*|_B} = \phi^*|_{\overline{B}}$. By Theorem 7.6 $(\phi^*|_B)^\# : A^\# \to B^\#$ is a homomorphism of K-algebras. Moreover, for $g \in G$ and $b \in B$,

$$\begin{aligned}(\phi^*|_B)^\# \circ e_G(g)(b) &= e_G(g)(\phi^*(b)) = \phi^*(b)(g) = b(\phi(g)) \\ &= e_H(\phi(g))(b) = e_H \circ \phi(g)(b).\end{aligned}$$

Hence $(\phi^*|_A)^\# \circ e_G = e_H \circ \phi$, i.e. the following diagram commutes:

$$\begin{tikzcd} KG \arrow[r, "\phi"] \arrow[d, "e_G"] & KH \arrow[d, "e_H"] \\ A^\# \arrow[r, "(\phi^*|_A)^\#"] & B^\# \end{tikzcd}$$

5.1 The action of the extended affine Weyl group on the affine Schur algebra

Recall that the extended affine Weyl group $\widehat{\Sigma}_n$ acts on $\widetilde{GL}_{n,a}$ by conjugation (see Lemma 3.4). In this way we regard $w \in \widehat{\Sigma}_n$ as an automorphism of $\widetilde{GL}_{n,a}$. Then the pull-back $w^* : K^{\widetilde{GL}_{n,a}} \to K^{\widetilde{GL}_{n,a}}$ is an algebra automorphism, and $(ww')^* = w'^* w^*$.

For $i, j \in \mathbb{Z}$ and $g \in \widetilde{GL}_{n,a}$ we have

$$w^*(c_{ij})(g) = c_{ij}(w(g)) = g'_{ij} = g_{w^{-1}(i),w^{-1}(j)} = c_{w^{-1}(i),w^{-1}(j)}(g)$$

Namely, $w^*(c_{ij}) = c_{w^{-1}i,w^{-1}j}$. Therefore for $\underline{i}, \underline{j} \in I(\mathbb{Z}, r)$ we have

$$w^*(c_{\underline{i},\underline{j}}) = c_{w^{-1}(\underline{i}),w^{-1}(\underline{j})}.$$

Let $f_w = w^*|_{\widetilde{A}(n,r)} : \widetilde{A}(n,r) \to \widetilde{A}(n,r)$ be the restriction of w^* to $\widetilde{A}(n,r)$. Then f_w is a homomorphism of formal coalgebras, f_w is surjective, and $\overline{f}_w = w^*|_{\overline{\widetilde{A}(n,r)}}$ is injective. Moreover, we have $f_w \circ f_{w'} = f_{w'w}$. Take the dual and we obtain an algebra automorphism

$$f_w^\# : \widetilde{S}(n,r) \to \widetilde{S}(n,r).$$

It follows by Corollary 7.3 that $f_{w'}^\# \circ f_w^\# = f_{w'w}^\#$.

Set $w(\xi) = f_w^\#(\xi)$. This defines an action of $\widehat{\Sigma}_n$ on the affine Schur algebra $\widetilde{S}(n,r)$. Precisely, for $\underline{i}, \underline{j}, \underline{p}, \underline{q} \in I(\mathbb{Z}, r)$ we have

$$\begin{aligned} w(\xi_{\underline{i},\underline{j}})(c_{\underline{p},\underline{q}}) &= \xi_{\underline{i},\underline{j}}(f_w(c_{\underline{p},\underline{q}})) = \xi_{\underline{i},\underline{j}}(c_{w^{-1}(\underline{p}),w^{-1}(\underline{q})}) \\ &= \begin{cases} 1, & \text{if } (\underline{i}, \underline{j}) \sim_{\widehat{\Sigma}_r} (w^{-1}(\underline{p}), w^{-1}(\underline{q})) \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} 1, & \text{if } (w(\underline{i}), w(\underline{j})) \sim_{\widehat{\Sigma}_r} (\underline{p}, \underline{q}) \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Therefore $w(\xi_{\underline{i},\underline{j}}) = \xi_{w(\underline{i}),w(\underline{j})}$. Thus we have

Proposition 5.1. *The $\widehat{\Sigma}_n$-action on $\widetilde{GL}_{n,a}$ induces an $\widehat{\Sigma}_n$-action on $\widetilde{S}(n,r)$: $w(\xi_{\underline{i},\underline{j}}) = \xi_{w(\underline{i}),w(\underline{j})}$ where $w \in \widehat{\Sigma}_n$, $\underline{i}, \underline{j} \in I(\mathbb{Z}, r)$. In particular, $\rho(\xi_{\underline{i},\underline{j}}) = \xi_{\underline{i}-(1\ldots1),\underline{j}-(1\ldots1)}$.* □

5.2 Endomorphisms of the affine Schur algebra

Let $a \in K^{\times}, s \in \mathbb{Z}\backslash\{0\}$. Recall that $\eta_{a,s} : \widetilde{GL}_{n,a} \to \widetilde{GL}_{n,1}$ is an injective semigroup homomorphism (see Lemma 3.2 (ii)). The pull-back $\eta^*_{a,s} : K^{\widetilde{GL}_{n,1}} \to K^{\widetilde{GL}_{n,a}}, \quad f \mapsto f \circ \eta_{a,s}$ is an algebra homomorphism. For $i, j = 1, \ldots, n$, $l \in \mathbb{Z}$ and $g \in \widetilde{GL}_{n,a}$, we have

$$\eta^*_{a,s}(c_{i,j+ln})(g) = c_{i,j+ln}(\eta_{a,s}(g)) = \begin{cases} a^{\frac{l}{s}} c_{i,j+\frac{l}{s}n}(g), & \text{if } s \mid l, \\ 0, & \text{otherwise.} \end{cases}$$

i.e.

$$\eta^*_{a,s}(c_{i,j+ln}) = \begin{cases} a^{\frac{l}{s}} c_{i,j+\frac{l}{s}n}, & \text{if } s \mid l, \\ 0, & \text{otherwise.} \end{cases}$$

For $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_r) \in \mathbb{Z}^r$ we define $\mathrm{ht}(\varepsilon) = \varepsilon_1 + \ldots + \varepsilon_r$. Then for $\underline{i}, \underline{j} \in I(n,r)$ and $\varepsilon \in \mathbb{Z}^r$ we have

$$\eta^*_{a,s}(c_{\underline{i},\underline{j}+n\varepsilon}) = \begin{cases} a^{\mathrm{ht}(\frac{\varepsilon}{s})} c_{\underline{i},\underline{j}+n\frac{\varepsilon}{s}}, & \text{if } s \mid \varepsilon_1, \ldots, \varepsilon_r, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\varphi_{a,s} = \eta^*_{a,s}|_{\widetilde{A}(n,r)} : \widetilde{A}(n,r) \to \widetilde{A}(n,r)$ be the restriction of $\eta^*_{a,s}$ to $\widetilde{A}(n,r)$. Then $\varphi_{a,s}$ is a surjective homomorphism of formal coalgebras. Taking the dual we obtain an injective algebra endomorphism

$$\psi_{a,s} = \varphi^{\#}_{a,s} : \widetilde{S}(n,r) \to \widetilde{S}(n,r).$$

Precisely, for $\underline{i}, \underline{j}, \underline{p}, \underline{q} \in I(n,r)$ and $\varepsilon, \varepsilon' \in \mathbb{Z}^r$, we have

$$\begin{aligned} \psi_{a,s}(\xi_{\underline{i},\underline{j}+n\varepsilon})(c_{\underline{p},\underline{q}+n\varepsilon'}) &= \xi_{\underline{i},\underline{j}+n\varepsilon}(\varphi_{a,s}(c_{\underline{p},\underline{q}+n\varepsilon'})) \\ &= \begin{cases} \xi_{\underline{i},\underline{j}+n\varepsilon}(a^{\mathrm{ht}(\frac{\varepsilon'}{s})} c_{\underline{p},\underline{q}+n\frac{\varepsilon'}{s}}), & \text{if } s \mid \varepsilon'_1, \ldots, \varepsilon'_r \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} a^{\mathrm{ht}(\varepsilon)}, & \text{if } s \mid \varepsilon'_1, \ldots, \varepsilon'_r, \text{ and } (\underline{i}, \underline{j} + n\varepsilon) \sim_{\Sigma_r} (\underline{p}, \underline{q} + n\frac{\varepsilon'}{s}) \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Therefore $\psi_{a,s}(\xi_{\underline{i},\underline{j}+n\varepsilon}) = a^{\mathrm{ht}(\varepsilon)} \xi_{\underline{i},\underline{j}+ns\varepsilon}$. It is easy to see that the restriction of $\psi_{a,s}$ to $S(n,r)$ is the identity map.

Recall that $\eta_a : \widetilde{GL}_{n,a} \to GL_n$ is a surjective semigroup homomorphism (see Lemma 3.2 (ii)). The pull-back $\eta^*_a : K^{GL_n} \to K^{\widetilde{GL}_{n,a}}, \quad f \mapsto f \circ \eta_a$ is an injective algebra homomorphism. For $i, j = 1, \ldots, n$ and $g \in \widetilde{GL}_{n,a}$ we have

$$\eta^*_a(c_{ij})(g) = c_{ij}(\eta_a(g)) = \sum_{l \in \mathbb{Z}} a^l g_{i,j+ln} = \sum_{l \in \mathbb{Z}} a^l c_{i,j+ln}(g)$$

i.e. $\eta^*_a(c_{ij}) = \sum_{l \in \mathbb{Z}} a^l c_{i,j+ln}$. Therefore for $\underline{i}, \underline{j} \in I(n,r)$ we have

$$\eta^*_a(c_{\underline{i},\underline{j}}) = \sum_{\varepsilon \in \mathbb{Z}^r} a^{\mathrm{ht}(\varepsilon)} c_{\underline{i},\underline{j}+n\varepsilon}.$$

Let $\varphi_a : A(n,r) \to \overline{\widetilde{A}(n,r)}$ denote the restriction of η_a^* to $A(n,r)$. Then φ_a is a formal homomorphism of formal coalgebras. Moreover $\overline{\varphi_a} = \varphi_a$ is injective. Taking the dual we obtain a surjective algebra homomorphism

$$\psi_a = \varphi_a^{\#} : \widetilde{S}(n,r) \to S(n,r).$$

Precisely, for $\underline{i}, \underline{j}, \underline{p}, \underline{q} \in I(n,r)$ and $\varepsilon \in \mathbb{Z}^r$, we have

$$\begin{array}{rcl} \psi_a(\xi_{\underline{i},\underline{j}+n\varepsilon})(c_{\underline{p},\underline{q}}) & = & \xi_{\underline{i},\underline{j}+n\varepsilon}(\varphi_a(c_{\underline{p},\underline{q}})) \\ & = & \xi_{\underline{i},\underline{j}+n\varepsilon}(\sum_{\varepsilon' \in \mathbb{Z}^r} a^{\mathrm{ht}(\varepsilon')} c_{\underline{p},\underline{q}+n\varepsilon'}) \\ & = & \sum_{\varepsilon' \in \mathbb{Z}^r} a^{\mathrm{ht}(\varepsilon')} \xi_{\underline{i},\underline{j}+n\varepsilon}(c_{\underline{p},\underline{q}+n\varepsilon'}) \\ & = & \begin{cases} a^{\mathrm{ht}(\varepsilon)}[\Sigma_{\underline{i},\underline{j}} : \Sigma_{\underline{i},\underline{j},\varepsilon}], & \text{if } (\underline{i},\underline{j}) \sim_{\Sigma_r} (\underline{p},\underline{q}) \\ 0, & \text{otherwise.} \end{cases} \end{array}$$

Therefore $\psi_a(\xi_{\underline{i},\underline{j}+n\varepsilon}) = a^{\mathrm{ht}(\varepsilon)}[\Sigma_{\underline{i},\underline{j}} : \Sigma_{\underline{i},\underline{j},\varepsilon}]\xi_{\underline{i},\underline{j}}$. It is easy to see that the restriction of ψ_a to $S(n,r)$ is the identity map.

In summary we have

Proposition 5.2. *Let $a \in K^\times$, $s \in \mathbb{Z}\backslash\{0\}$.*

(i) *The injective semigroup homomorphism $\eta_{a,s}$ induces an injective algebra endomorphism of the affine Schur algebra*

$$\psi_{a,s} : \widetilde{S}(n,r) \to \widetilde{S}(n,r), \quad \xi_{\underline{i},\underline{j}+n\varepsilon} \mapsto a^{\mathrm{ht}(\varepsilon)}\xi_{\underline{i},\underline{j}+ns\varepsilon}$$

where $\underline{i}, \underline{j} \in I(n,r)$ and $\varepsilon \in \mathbb{Z}^r$.

(ii) *The surjective semigroup homomorphism η_a induces a surjective algebra homomorphism*

$$\psi_a : \widetilde{S}(n,r) \to S(n,r), \quad \xi_{\underline{i},\underline{j}+n\varepsilon} \mapsto a^{\mathrm{ht}(\varepsilon)}[\Sigma_{\underline{i},\underline{j}} : \Sigma_{\underline{i},\underline{j},\varepsilon}]\xi_{\underline{i},\underline{j}}$$

where $\underline{i}, \underline{j} \in I(n,r)$ and $\varepsilon \in \mathbb{Z}^r$.

(iii) *The restrictions of $\psi_{a,s}$ and ψ_a to $S(n,r)$ are the identity map.* □

Denote by $\psi_{a,0}$ the composition $\widetilde{S}(n,r) \xrightarrow{\psi_a} S(n,r) \hookrightarrow \widetilde{S}(n,r)$. Then for any $s \in \mathbb{Z}$ we have $\psi_{a,s}(\xi_{\underline{i},\underline{j}+n\varepsilon}) = a^{\mathrm{ht}(\varepsilon)}[\Sigma_{\underline{i},\underline{j}} : \Sigma_{\underline{i},\underline{j},\varepsilon}]^{\delta_{s,0}}\xi_{\underline{i},\underline{j}+ns\varepsilon}$.

Proposition 5.3. *Assume $a, a' \in K^\times$ and $s, s' \in \mathbb{Z}$. Then $\psi_{a,s} \circ \psi_{a',s'} = \psi_{a'a^{s'},ss'}$.*

Proof. For $\underline{i}, \underline{j} \in I(n,r)$ and $\varepsilon \in \mathbb{Z}^r$ we have

$$\begin{array}{rcl} \psi_{a,s} \circ \psi_{a',s'}(\xi_{\underline{i},\underline{j}+n\varepsilon}) & = & \psi_{a,s}((a')^{\mathrm{ht}(\varepsilon)}[\Sigma_{\underline{i},\underline{j}} : \Sigma_{\underline{i},\underline{j},\varepsilon}]^{\delta_{s',0}}\xi_{\underline{i},\underline{j}+ns'\varepsilon}) \\ & = & (a')^{\mathrm{ht}(\varepsilon)}[\Sigma_{\underline{i},\underline{j}} : \Sigma_{\underline{i},\underline{j},\varepsilon}]^{\delta_{s',0}} a^{\mathrm{ht}(s'\varepsilon)}[\Sigma_{\underline{i},\underline{j}} : \Sigma_{\underline{i},\underline{j},s'\varepsilon}]^{\delta_{s,0}}\xi_{\underline{i},\underline{j}+nss'\varepsilon} \\ & = & (a'a^{s'})^{\mathrm{ht}(\varepsilon)}[\Sigma_{\underline{i},\underline{j}} : \Sigma_{\underline{i},\underline{j},\varepsilon}]^{\delta_{ss',0}}\xi_{\underline{i},\underline{j}+nss'\varepsilon} \\ & = & \psi_{a'a^{s'},ss'}(\xi_{\underline{i},\underline{j}+n\varepsilon}). \end{array}$$

Therefore $\psi_{a,s} \circ \psi_{a',s'} = \psi_{a'a^{s'},ss'}$. □

The transpose g^{tr} of a matrix $g \in \mathfrak{M}_n$ is also a matrix in $\mathfrak{M}_n$. It follows by Lemma 3.1 (ii) that

$$\widetilde{\det}_{a^{-1}}(g^{tr}) = \det(\eta_{a^{-1}}(g^{tr})) = \det(\eta_a(g)^{tr}) = \det(\eta_a(g)) = \widetilde{\det}_a(g).$$

Therefore $g \in \widetilde{GL}_{n,a}$ if and only if $g^{tr} \in \widetilde{GL}_{n,a^{-1}}$. In particular, the map $T : g \mapsto g^{tr}$ is a semigroup anti-isomorphism from $\widetilde{GL}_{n,a}$ to $\widetilde{GL}_{n,a^{-1}}$. Therefore taking transpose induces an algebra isomorphism

$$T^* : K^{\widetilde{GL}_{n,a^{-1}}} \to K^{\widetilde{GL}_{n,a}}, \quad f \mapsto f \circ T.$$

For $i, j \in \mathbb{Z}$ and $g \in \widetilde{GL}_{n,a}$ we have $T^*(c_{ij})(g) = c_{ij}(g^{tr}) = c_{ji}(g)$, i.e. $T^*(c_{ij}) = c_{ji}$. Therefore for $\underline{i}, \underline{j} \in I(\mathbb{Z}, r)$ we have

$$T^*(c_{\underline{i},\underline{j}}) = c_{\underline{j},\underline{i}}.$$

The restriction map $T^*|_{\widetilde{A}(n,r)} : \widetilde{A}(n,r) \to \widetilde{A}(n,r)$ is an anti-homomorphism of formal coalgebras. Moreover, the map $\overline{T^*|_{\widetilde{A}(n,r)}} = T^*|_{\overline{\widetilde{A}(n,r)}}$ is injective and $T^*|_{\widetilde{A}(n,r)}$ is surjective. Take the dual and we obtain an algebra anti-automorphism

$$\widetilde{J} = (T^*|_{\widetilde{A}(n,r)})^{\#} : \widetilde{S}(n,r) \to \widetilde{S}(n,r), \quad \xi_{\underline{i},\underline{j}} \mapsto \xi_{\underline{j},\underline{i}}$$

where $\underline{i}, \underline{j} \in I(\mathbb{Z}, r)$. The quantized $\widetilde{J}$ is given in [17] Lemma 1.11.

5.3 Transfer maps

The multiplicative function $\widetilde{\det}_a$ on $\widetilde{GL}_{n,a}$ can be written as

$$\widetilde{\det}_a = \sum_{\sigma \in \Sigma_n} \operatorname{sgn}(\sigma) \sum_{\varepsilon \in \mathbb{Z}^n} a^{\operatorname{ht}(\varepsilon)} c_{(12\ldots n),(12\ldots n)\sigma + n\varepsilon} \in \overline{\widetilde{A}(n,n)}.$$

For $g, h \in \widetilde{GL}_{n,a}$ we have

$$\Delta(\widetilde{\det}_a)(g,h) = \widetilde{\det}_a(gh) = \widetilde{\det}_a(g)\widetilde{\det}_a(h) = (\widetilde{\det}_a \otimes \widetilde{\det}_a)(g,h).$$

So $\Delta(\widetilde{\det}_a) = \widetilde{\det}_a \otimes \widetilde{\det}_a$. Therefore multiplying by $\widetilde{\det}_a$ is a formal homomorphism $\widetilde{\det}_a : \widetilde{A}(n,r) \to \widetilde{A}(n, n+r)$ of formal coalgebras.

Lemma 5.4. *We have* $\widetilde{\det}_a \circ \varphi_{a,1} = \overline{\varphi_{a,1}} \circ \widetilde{\det}_1$.

Proof. Note that $\widetilde{\det}_a = \overline{\varphi_{a,1}}(\widetilde{\det}_1)$. For $c \in \widetilde{A}(n,r)$ we have

$$\begin{aligned}\widetilde{\det}_a \circ \varphi_{a,1}(c) &= \varphi_{a,1}(c)\widetilde{\det}_a = \varphi_{a,1}(c)\overline{\varphi_{a,1}}(\widetilde{\det}_1) \\ &= \overline{\varphi_{a,1}}(c\ \widetilde{\det}_1) = \overline{\varphi_{a,1}} \circ \widetilde{\det}_1(c).\end{aligned}$$

Therefore $\widetilde{\det}_a \circ \varphi_{a,1} = \overline{\varphi_{a,1}} \circ \widetilde{\det}_1$. $\square$

It follows by Proposition 3.6 that the map $\overline{\widetilde{\det}_a}$ is injective. Take the dual of the map $\widetilde{\det}_a$ and we obtain a surjective algebra homomorphism

$$\widetilde{\det}_a^{\#} : \widetilde{S}(n, n+r) \to \widetilde{S}(n,r), \quad \xi_{\underline{i},\underline{j}} \mapsto \Big(c \mapsto \xi_{\underline{i},\underline{j}}(c\ \widetilde{\det}_a) \text{ for } c \in \widetilde{A}(n,r)\Big).$$

Proposition 5.5. (i) *We have* $\psi_{a,1} \circ \widetilde{\det}_a^{\#} = \widetilde{\det}_1^{\#} \circ \psi_{a,1}$.

(ii) *We have* $\widetilde{\det}_a|_{S(n,n+r)} = \det^*$.

(iii) *On* $K\widetilde{SL}_{n,a}$ *we have* $\widetilde{e}_a^{\,r} = \widetilde{\det}_a^{\#} \circ \widetilde{e}_a^{\,n+r}$.

Proof. (i) This follows from Lemma 5.4 and Theorem 7.2 (v).

(ii) Let $\underline{i}, \underline{j} \in I(n, n+r)$, $\underline{p}, \underline{q} \in I(n, r)$. Then

$$\begin{aligned}
\widetilde{\det}_a|_{S(n,n+r)}(\xi_{\underline{i},\underline{j}})(c_{\underline{p},\underline{q}}) &= \xi_{\underline{i},\underline{j}}(\widetilde{\det}_a c_{\underline{p},\underline{q}}) \\
&= \xi_{\underline{i},\underline{j}}\big(\textstyle\sum_{\sigma\in\Sigma_n} \operatorname{sgn}(\sigma) \sum_{\varepsilon\in\mathbb{Z}^n} a^{\mathrm{ht}(\varepsilon)} c_{(12\ldots n),(12\ldots n)\sigma+n\varepsilon} c_{\underline{p},\underline{q}}\big) \\
&= \textstyle\sum_{\sigma\in\Sigma_n} \operatorname{sgn}(\sigma) \sum_{\varepsilon\in\mathbb{Z}^n} a^{\mathrm{ht}(\varepsilon)} \xi_{\underline{i},\underline{j}}(c_{(12\ldots n),(12\ldots n)\sigma+n\varepsilon} c_{\underline{p},\underline{q}}).
\end{aligned}$$

Since $\underline{i}, \underline{j} \in I(n, n+r), \underline{p}, \underline{q} \in I(n, r)$, we have $\varepsilon \neq 0$ implies $\xi_{\underline{i},\underline{j}}(c_{(12\ldots n),(12\ldots n)\sigma+n\varepsilon} c_{\underline{p},\underline{q}}) = 0$. Therefore

$$\begin{aligned}
\widetilde{\det}_a|_{S(n,n+r)}(\xi_{\underline{i},\underline{j}})(c_{\underline{p},\underline{q}}) &= \textstyle\sum_{\sigma\in\Sigma_n} \operatorname{sgn}(\sigma) \xi_{\underline{i},\underline{j}}(c_{(12\ldots n),(12\ldots n)\sigma} c_{\underline{p},\underline{q}}) \\
&= \xi_{\underline{i},\underline{j}}\big(\textstyle\sum_{\sigma\in\Sigma_n} \operatorname{sgn}(\sigma) c_{(12\ldots n),(12\ldots n)\sigma} c_{\underline{p},\underline{q}}\big) \\
&= \xi_{\underline{i},\underline{j}}(\det\, c_{\underline{p},\underline{q}}) = \det^*(\xi_{\underline{i},\underline{j}})(c_{\underline{p},\underline{q}}).
\end{aligned}$$

(iii) For $g \in \widetilde{SL}_{n,a}$, $c \in \widetilde{A}(n, r)$,

$$\begin{aligned}
\widetilde{\det}_a^{\#} \circ \widetilde{e}_a^{\,n+r}(g)(c) &= \widetilde{e}_a^{\,n+r}(g)(c\ \widetilde{\det}_a) = (c\ \widetilde{\det}_a)(g) \\
&= c(g)\widetilde{\det}_a(g) = c(g) = \widetilde{e}_a^{\,r}(g)(c).
\end{aligned}$$

□

Theorem 5.6. *The following diagram commutes:*

$$\begin{tikzcd}
& KSL_n \arrow[rr, hook] \arrow[dl, two heads, "e^{n+r}"'] \arrow[dr, two heads, "e^r"] & & KGL_n \arrow[dl] \\
S(n,n+r) \arrow[rr, two heads, "\det^*"'] & & S(n,r) & \\
& K\widetilde{SL}_{n,a} \arrow[uu, dashed, two heads, "\eta_a"] \arrow[rr, dashed, hook] \arrow[dl, dashed, two heads, "\widetilde{e}_a^{\,n+r}"'] \arrow[dr, dashed, two heads, "\widetilde{e}_a^{\,r}"] & & K\widetilde{GL}_{n,a} \arrow[uu, two heads, "\eta_a"'] \arrow[dl] \\
\widetilde{S}(n,n+r) \arrow[uu, two heads, "\psi_a"] \arrow[rr, two heads, "\widetilde{\det}_a^{\#}"'] & & \widetilde{S}(n,r) \arrow[uu, two heads, "\psi_a"] &
\end{tikzcd}$$

Proof. By Proposition 5.5 it suffices to prove $\psi_a \circ \widetilde{\det}_a^{\#} = \det^* \circ \psi_a$. Following the commutative parts of the diagram we have

$$\begin{aligned}
\psi_a \circ \widetilde{\det}_a^{\#} \circ \widetilde{e}_a^{\,n+r} &= \psi_a \circ \widetilde{e}_a^{\,r} = e^r \circ \eta_a \\
&= \det^* \circ e^{n+r} \circ \eta_a = \det^* \circ \psi_a \circ \widetilde{e}_a^{\,n+r}.
\end{aligned}$$

It follows from the surjectivity of $\widetilde{e}_a^{\,n+r}$ that $\psi_a \circ \widetilde{\det}_a^{\#} = \det^* \circ \psi_a$. □

6 The Lie algebras $\mathfrak{g}_n$ and $\mathfrak{g}'_n$

In this section, $K = \mathbb{C}$, and $n \geq 2$.

Let $\mathfrak{g}_n$ be the underlying Lie algebra of $\mathfrak{M}_n$ with Lie bracket the commutator, and $\mathfrak{g}'_n = [\mathfrak{g}_n, \mathfrak{g}_n]$ its Lie subalgebra. Then $\mathfrak{g}_n = \mathfrak{gl}_n[t, t^{-1}]$ and $\mathfrak{g}'_n = \mathfrak{sl}_n[t, t^{-1}]$ are loop algebras, and they are quotients of the affine general linear Lie algebra $\widehat{\mathfrak{gl}}_n$ and the affine special linear Lie algebra $\widehat{\mathfrak{sl}}_n$ respectively.

Lemma 6.1. *([14]) As a Lie algebra over $\mathbb{C}$, the loop algebra $\mathfrak{g}'_n$ is generated by $\{E_{s,s+1}, E_{s,s-1} | s = 1, \ldots, n\}$.* □

The universal enveloping algebra $\mathcal{U}(\mathfrak{g}_n)$ of $\mathfrak{g}_n$ acts naturally on $\widetilde{E}$, and hence on $\widetilde{E}^{\otimes r}$ via the comultiplication. Let $\widetilde{\pi} : \mathcal{U}(\mathfrak{g}_n) \to \mathrm{End}_{\mathbb{C}}(\widetilde{E}^{\otimes r})$ be the corresponding representation map.

Lemma 6.2. *This action commutes with the right action of $\widehat{\Sigma}_r$. In particular, the image $\mathrm{Im}(\widetilde{\pi})$ of $\widetilde{\pi}$ is a subalgebra of $\widetilde{S}(n, r)$.*

Proof. Let $s \in \{1, \ldots, n\}$, $t \in \mathbb{Z}$. For $\underline{i} \in I(\mathbb{Z}, r)$ and $k = 1, \ldots, r$ we define $\underline{i}^k = (i_1, \ldots, i_{k-1}, s - t + i_k, i_{k+1}, \ldots, i_r)$.

Let $\underline{i} \in I(\mathbb{Z}, r)$, and $w = (\sigma, \varepsilon) \in \widehat{\Sigma}_r$ with $\sigma \in \Sigma_r$, $\varepsilon \in \mathbb{Z}^r$. We have

$$\begin{aligned}
(E_{st} v_{\underline{i}})w &= ((\textstyle\sum_{k=1}^r 1^{\otimes k-1} \otimes E_{st} \otimes 1^{\otimes r-k}) v_{\underline{i}})w \\
&= (\textstyle\sum_{k=1}^r \delta_{\bar{t}, \bar{i}_k} v_{\underline{i}^k})w = \textstyle\sum_{k=1}^r \delta_{\bar{t}, \bar{i}_k} v_{\underline{i}^k w}
\end{aligned}$$

$$\begin{aligned}
E_{st}(v_{\underline{i}} w) = E_{st}(v_{\underline{i}w}) &= \textstyle\sum_{k=1}^r \delta_{\bar{t}, \overline{(\underline{i}w)}_k} v_{(\underline{i}w)^k} \\
&= \textstyle\sum_{k=1}^r \delta_{\bar{t}, \overline{(\underline{i}\sigma)}_k} v_{(\underline{i}\sigma + n\varepsilon)^k} = \textstyle\sum_{k=1}^r \delta_{\bar{t}, \bar{i}_{\sigma^{-1}(k)}} v_{(\underline{i}\sigma)^k + n\varepsilon} \\
&= \textstyle\sum_{k=1}^r \delta_{\bar{t}, \bar{i}_{\sigma^{-1}(k)}} v_{\underline{i}^{\sigma^{-1}(k)}\sigma + n\varepsilon} \\
&= \textstyle\sum_{k=1}^r \delta_{\bar{t}, \bar{i}_k} v_{\underline{i}^k \sigma + n\varepsilon} = \textstyle\sum_{k=1}^r \delta_{\bar{t}, \bar{i}_k} v_{\underline{i}^k w}.
\end{aligned}$$

This completes the proof. □

Lemma 6.3. *Let $s = 1, \ldots, n$, $t \in \mathbb{Z}$, $s \neq t$. Then*

$$\begin{aligned}
\widetilde{\pi}(E_{st}) &= \textstyle\sum_{\underline{i} \in I(n, r-1)/\Sigma_{r-1}} \xi_{\underline{i}s, \underline{i}t} \\
\widetilde{\pi}(E_{ss}) &= \textstyle\sum_{\underline{i} \in I(n, r)/\Sigma_r} \lambda_{\underline{i}, s} \xi_{\underline{i}, \underline{i}}
\end{aligned}$$

where $\lambda_{\underline{i}, s}$ is the number of s in $\underline{i}$.

Proof. Let $\underline{q} \in I(\mathbb{Z}, r)$. We have

$$\begin{aligned}
(\textstyle\sum_{\underline{i}} \xi_{\underline{i}s, \underline{i}t})(v_{\underline{q}}) &= \textstyle\sum_{\underline{i}} \xi_{\underline{i}s, \underline{i}t}(v_{\underline{q}}) = \textstyle\sum_{\underline{i}} \sum_{\underline{p} \in I(\mathbb{Z}, r)} \xi_{\underline{i}s, \underline{i}t}(c_{\underline{p}, \underline{q}}) v_{\underline{p}} \\
&= \textstyle\sum_{\underline{i}} \sum_{k=1}^r \xi_{\underline{i}s, \underline{i}t}(c_{\underline{q}^k, \underline{q}}) v_{\underline{q}^k} = \textstyle\sum_{k=1}^r \delta_{\bar{t}, \bar{q}_k} v_{\underline{q}^k} \\
&= E_{st}(v_{\underline{q}})
\end{aligned}$$

$$\begin{aligned}
(\textstyle\sum_{\underline{i}} \lambda_{\underline{i}, s} \xi_{\underline{i}, \underline{i}})(v_{\underline{q}}) &= \textstyle\sum_{\underline{i}} \lambda_{\underline{i}, s} \xi_{\underline{i}, \underline{i}}(v_{\underline{q}}) = \lambda_{\underline{q}} v_{\underline{q}} = \textstyle\sum_{k=1}^r \delta_{s, \bar{q}_k} v_{\underline{q}^k} \\
&= E_{ss}(v_{\underline{q}})
\end{aligned}$$

as desired. □

Lemma 6.4. *The restriction map $\widetilde{\pi}|_{\mathcal{U}(\mathfrak{gl}_n)}$ equals to π (defined in Section 2.4).*

Proof. It is enough to prove that if $s, t = 1, \dots, n$, $s \neq t$, then

$$\begin{aligned} \pi(E_{st}) &= \textstyle\sum_{\underline{i} \in I(n,r-1)/\Sigma_{r-1}} \xi_{\underline{i}s,\underline{i}t} \\ \pi(E_{ss}) &= \textstyle\sum_{\underline{i} \in I(n,r)/\Sigma_r} \lambda_{\underline{i},s} \xi_{\underline{i},\underline{i}}. \end{aligned}$$

The proof is the same as that of Lemma 6.3 except that we need to replace $\underline{q} \in I(\mathbb{Z}, r)$ by $\underline{q} \in I(n, r)$. □

By Lemma 6.4 $\widetilde{\pi}(\mathcal{U}(\mathfrak{sl}_n)) = \pi(\mathcal{U}(\mathfrak{sl}_n)) = S(n,r)$ contains all $\xi_{\underline{i},\underline{i}}$, $\underline{i} \in I(n,r)$. Therefore by Lemma 6.3 $\widetilde{\pi}(\mathcal{U}(\mathfrak{g}_n))$ contains a subalgebra of $\widetilde{S}(n,r)$ generated by Y and $\widetilde{\pi}(\mathcal{U}(\mathfrak{g}'_n))$ contains a subalgebra of $\widetilde{S}(n,r)$ generated by $X = X_1 \cup X_2$, where

$$\begin{aligned} Y &= \{\xi_{\underline{i}s,\underline{i}t} | \underline{i} \in I(n, r-1), s = 1, \dots, n, t \in \mathbb{Z}\} \\ X_1 &= \{\xi_{\underline{i}s,\underline{i}(s+1)} | \underline{i} \in I(n, r-1), s = 1, \dots, n\} \\ X_2 &= \{\xi_{\underline{i}s,\underline{i}(s-1)} | \underline{i} \in I(n, r-1), s = 1, \dots, n\}. \end{aligned}$$

Lemma 6.5. (i) *As a K-algebra, $\widetilde{S}(n,r)$ is generated by Y.*

(ii) *Assume $r < n$. As a K-algebra, $\widetilde{S}(n,r)$ is generated by X.*

Proof. (i) For $\xi_{\underline{i},\underline{j}} \in \widetilde{S}(n,r)$ with $\underline{i} \in I(n,r)$, define its index to be the number of s in $\{1, \dots, r\}$ such that $j_s \neq i_s$. Induct on the index. Clearly Y is the set of all $\xi_{\underline{i},\underline{j}}$'s of index 0 and 1.

Suppose $\xi_{\underline{i},\underline{j}}$ is of index $m \geq 2$. Without loss of generality we may assume that $i_1 \neq j_1$. Assume $1 \leq p \leq q \leq r$ are such that

$$\begin{aligned} &i_s = j_s = i_1 \text{ for } 2 \leq s \leq p, \\ &j_s \equiv i_1 \pmod{n},\ j_s \neq i_s \text{ for } p+1 \leq s \leq q, \\ &j_s \not\equiv i_1 \pmod{n} \text{ for } q+1 \leq s \leq r. \end{aligned}$$

Let $\underline{j}' = i_1 j_2 \dots j_r$. Then

$$\xi_{\underline{i},\underline{j}'} \xi_{\underline{j}',\underline{j}} = a_{\underline{i},\underline{j}} \xi_{\underline{i},\underline{j}} + \sum_{\underline{k}} a_{\underline{i},\underline{k}} \xi_{\underline{i},\underline{k}}$$

where $\underline{k}$ lies in the set $\{(i_1 \dots i_1 j_{p+1} \dots j_{s-1}(j_s + j_1 - i_1) j_{s+1} \dots j_r) | p+1 \leq s \leq q\}$ and $a_{\underline{i},\underline{j}} \neq 0$.

In the above equality, the elements $\xi_{\underline{i},\underline{j}'}$, $\xi_{\underline{j}',\underline{j}}$, and all $\xi_{\underline{i},\underline{k}}$ have indices smaller than m. So by induction we can finish the proof.

(ii) We only need to show X generates Y. Let $s \in \{1, \dots, n\}$, $t \in \mathbb{Z}$. We may assume that $s \leq t$.

Note that by Proposition 5.1 the action of $\rho \in \widehat{\Sigma}_n$ on $\widetilde{S}(n,r)$ permutes X_1. So if $s \leq t < s+n$, then $\rho^{-s+1}(\xi_{\underline{i}s,\underline{i}t}) = \xi_{\underline{i}'1,\underline{i}'(t-s+1)}$ lies in $S(n,r)$, where $\underline{i}' = \overline{\underline{i} - (s-1)(1\dots1)} \in I(n, r-1)$. Now $\xi_{\underline{i}'1,\underline{i}'(t-s+1)}$ is generated by elements in $X_1 \cap S(n,r)$, and hence $\xi_{\underline{i}s,\underline{i}t}$ is generated by elements in $\rho^{s-1}(X_1 \cap S(n,r)) \subseteq X_1$.

If $t \geq s+n$, then there exists s' such that $s < s' < s+n$ and

$$s' \not\equiv i_p \pmod{n} \text{ for any } 1 \leq p \leq r-1.$$

Therefore

$$\xi_{\underline{i}s,\underline{i}s'}\xi_{\underline{i}s',\underline{i}t} = \xi_{\underline{i}s,\underline{i}t}.$$

By induction, we get the desired result. $\square$

Remark. Note that the coefficients $a_{\underline{i},\underline{j}}$ and $a_{\underline{i},\underline{k}}$ in the proof of Lemma 6.5 are smaller than $r!$. So Lemma 6.5 holds over K whose characteristic equals to 0 or is greater than $r!$. $\square$

Theorem 6.6. (i) *The natural action of $\mathcal{U}(\mathfrak{g}_n)$ on $\widetilde{E}^{\otimes r}$ induces a surjection $\widetilde{\pi} : \mathcal{U}(\mathfrak{g}_n) \to \widetilde{S}(n, r)$.*

(ii) *On $\mathcal{U}(\mathfrak{g}'_n)$, we have $\widetilde{\pi}^r = \widetilde{\det}_a^{\#} \circ \widetilde{\pi}^{n+r}$. Moreover, the following diagram commutes:*

$$\begin{tikzcd}
& \mathcal{U}(\mathfrak{sl}_n) \arrow[dl, two heads, "\pi^{n+r}"'] \arrow[dr, two heads, "\pi^r"] \arrow[rr, hook] & & \mathcal{U}(\mathfrak{gl}_n) \arrow[dl, two heads] \\
S(n, n+r) \arrow[rr, two heads, "\det^*"'] & & S(n, r) & \\
& \mathcal{U}(\mathfrak{g}'_n) \arrow[uu, dashed, two heads, "\eta_a"] \arrow[dl, dashed, "\widetilde{\pi}^{n+r}"'] \arrow[dr, dashed, "\widetilde{\pi}^r"] \arrow[rr, hook, dashed] & & \mathcal{U}(\mathfrak{g}_n) \arrow[uu, two heads, "\eta_a"'] \arrow[dl, two heads] \\
\widetilde{S}(n, n+r) \arrow[uu, two heads, "\psi_a"] \arrow[rr, two heads, "\widetilde{\det}_a^{\#}"'] & & \widetilde{S}(n, r) \arrow[uu, two heads, "\psi_a"] &
\end{tikzcd}$$

where η_a is induced from the map denoted by the same symbol defined in Section 3.

(iii) *When $r < n$, the restriction map $\widetilde{\pi} = \widetilde{\pi}|_{\mathcal{U}(\mathfrak{g}'_n)} : \mathcal{U}(\mathfrak{g}'_n) \to \widetilde{S}(n, r)$ is surjective.*

Proof. (i) It follows by Lemma 6.5 (i).

(ii) To prove the first assertion it suffices to prove the two maps coincide on a set of generators of $\mathcal{U}(\mathfrak{g}'_n)$. For $s = 1, \ldots, n$, $t = s \pm 1$,

$$\begin{aligned}
\widetilde{\det}_a^{\#} \circ \widetilde{\pi}^{n+r}(E_{st}) &= \widetilde{\det}_a^{\#}(\textstyle\sum_{\underline{p}\in I(n,n+r-1)/\Sigma_{n+r-1}} \xi_{\underline{p}s,\underline{p}t}) \\
&= \textstyle\sum_{\underline{p}\in I(n,n+r-1)/\Sigma_{n+r-1}} \widetilde{\det}_a^{\#}(\xi_{\underline{p}s,\underline{p}t}).
\end{aligned}$$

If $\{p_1, \ldots, p_{n+r-1}\} \not\supseteq \{1, \ldots, n\}$, then $\widetilde{\det}_a^{\#}(\xi_{\underline{p}s,\underline{p}t}) = 0$. If $\underline{p} \sim_{\Sigma_{n+r-1}} (1 \ldots n)\underline{q}$ for some $\underline{q} \in I(n, r-1)$, then $\widetilde{\det}_a^{\#}(\xi_{\underline{p}s,\underline{p}t}) = \xi_{\underline{q}s,\underline{q}t}$. Hence

$$\widetilde{\det}_a^{\#} \circ \widetilde{\pi}^{n+r}(E_{st}) = \sum_{\underline{q}\in I(n,r-1)/\Sigma_{r-1}} \xi_{\underline{q}s,\underline{q}t} = \widetilde{\pi}^r(E_{st}).$$

Concerning the commutativity of the diagram, by Section 2.4 and Theorem 5.6 it remains to prove $\psi_a \circ \widetilde{\pi}^{n+r}(E_{st}) = \pi^{n+r} \circ \eta_a(E_{st})$ for $s = 1, \ldots, n$, $t \in \mathbb{Z}$. A direct check completes the proof: if $s \neq t$, then by Lemma 6.3, Proposition 5.3 (ii), Lemma 6.4 and the definition of η_a we have

$$\begin{aligned}
\psi_a \circ \widetilde{\pi}^{n+r}(E_{st}) &= \psi_a(\textstyle\sum_{\underline{p}\in I(n,n+r-1)/\Sigma_{n+r-1}} \xi_{\underline{p}s,\underline{p}t}) \\
&= \textstyle\sum_{\underline{p}\in I(n,n+r-1)/\Sigma_{n+r-1}} a^{\frac{t-\bar{t}}{n}} \xi_{\underline{p}s,\underline{p}\bar{t}} \\
&= \pi^{n+r}(a^{\frac{t-\bar{t}}{n}} E_{s,\bar{t}}) = \pi^{n+r} \circ \eta_a(E_{st});
\end{aligned}$$

if $s = t$, then by Lemma 6.3 $\widetilde{\pi}^{n+r}(E_{ss})$ lies in $S(n,r)$, and hence by Proposition 5.3 (iii) and Lemma 3.1 (iii) we have $\psi_a \circ \widetilde{\pi}^{n+r}(E_{ss}) = \widetilde{\pi}^{n+r}(E_{ss}) = \pi^{n+r}(E_{ss}) = \pi^{n+r} \circ \eta_a(E_{ss})$.

(iii) It follows from Lemma 6.5 (ii). $\square$

As a consequence, we have

Theorem 6.7. *There is a surjective algebra homomorphism from $\mathcal{U}(\widehat{\mathfrak{gl}}_n)$ to $\widetilde{S}(n,r)$. When $r < n$, the restriction of this homomorphism to $\mathcal{U}(\widehat{\mathfrak{sl}}_n)$ is also surjective.*

Remark. G.Lusztig [17] showed that when $r \geq n$ the quantized $\widetilde{\pi}$ is not surjective. $\square$

7 Appendix 1 : Formal coalgebras

Let K be a field. In this section, when we say K-vector space we mean a K-vector space V with a fixed basis $\{v_i | i \in I\}$. Let $\overline{V}$ denote the closure of V with respect to formal sums, i.e. $\overline{V} = \{\sum_{i\in I} \lambda_i v_i | \lambda_i \in K\}$, and i_V the canonical embedding from V to $\overline{V}$. For example, the closure of the polynomial ring $K[X]$ with basis $\{X^i\}_{i\geq 0}$ is $K[[X]]$, the ring of formal power series. Another example is, if V is a finite dimensional space, then $\overline{V} = V$.

Denote by V^* the dual of V, i.e. $V^* = \{K\text{-linear maps } f : V \to K\}$, and by $V^\#$, called *the dual of V with finite support*, the subspace of V^* with basis $\{v_i^* | i \in I\}$ where $v_i^*(v_j) = \delta_{ij}$. Then $V^* = \overline{V^\#}$. Note that $V^* = V^\#$ if V is finite dimensional.

Let V, W be two K-vector spaces with bases $\{v_i | i \in I\}$ and $\{w_j | j \in J\}$ respectively. A K-linear map $f : V \to \overline{W}$ is called a *formal map* from V to W. Such a map corresponds to a $J \times I$ matrix with entries in K, say M. We say f *is row finite* if there are only finitely many nonzero entries in each row of M; we say f *is column finite* if there are only finitely many nonzero entries in each column of M. In fact, a formal map f is column finite means exactly the image $\mathrm{Im}(f)$ of f lies in W. A linear function $f \in V^*$ is row finite if and only if $f \in V^\#$.

The tensor product of V and W over K, denoted by $V \otimes W$, has a natural basis $\{v_i \otimes w_j | i \in I, j \in J\}$. The space $\overline{V} \otimes W$ is considered as a subspace of $\overline{V} \otimes \overline{W}$ via $(\sum_{i\in I} \mu_i v_i) \otimes w_j \mapsto \sum_{i\in I} \mu_i v_i \otimes w_j$, and the space $\overline{V} \otimes \overline{W}$ is considered as a subspace of $\overline{V \otimes W}$ via $(\sum_{i\in I} \mu_i v_i) \otimes (\sum_{j\in J} \lambda_j w_j) \mapsto \sum_{i\in I, j\in J} \mu_i \lambda_j v_i \otimes w_j$.

Proposition 7.1. *We have a canonical isomorphism $V^\# \otimes W^\# \to (V \otimes W)^\#$, $v_i^* \otimes w_j^* \mapsto (v_i \otimes w_j)^*$. We identify these two spaces.* $\square$

We have $V \cong (V^\#)^\#$ canonically. Identifying these two spaces, we have $\overline{V} = (V^\#)^*$. Thus, as a consequence of Proposition 7.1, we have $\overline{V \otimes V} = ((V \otimes V)^\#)^* = (V^\# \otimes V^\#)^*$.

Let $f : V \to \overline{W}$, $v_i \mapsto \sum_{j\in J} m_{ji} w_j$, be a row finite formal map. Then we can extend f to a linear map $\overline{f} : \overline{V} \to \overline{W}$, $\sum_{i\in I} \lambda_i v_i \mapsto \sum_{j\in j} (\sum_{i\in I} \lambda_i m_{ji}) w_j$.

Theorem 7.2. *Let V, W be two K-vector spaces with basis $\{v_i | i \in I\}$ and $\{w_j | j \in J\}$ respectively, and $f : V \to \overline{W}$ a formal map. Then $f^\# : W^\# \to V^*$, $\alpha \mapsto \overline{\alpha} \circ f$ is a formal map from $W^\#$ to $V^\#$. Moreover,*

(i) *If f is column finite, then $f^\#$ is row finite.*

(ii) *If f is row finite, then $f^\#$ is column finite.*

(iii) *If* $\operatorname{Im}(f) \supseteq W$*, then* $f^{\#}$ *is injective.*

(iv) *If* f *is row finite and* $\overline{f}$ *is injective, then* $\operatorname{Im}(f^{\#}) = V^{\#}$*.*

Let $g : W \to \overline{U}$ *be a row finite formal map.*

(v) *If* f *is also row finite, then* $\overline{g} \circ f$ *is row finite and* $\overline{\overline{g} \circ f} = \overline{g} \circ \overline{f} : \overline{V} \to \overline{U}$*.*

(vi) *We have* $(\overline{g} \circ f)^{\#} = f^{\#} \circ g^{\#} : U^{\#} \to V^{*}$*.*

Proof. Let $M = (m_{ji})_{i \in I, j \in J}$ be the matrix corresponding to f. Then

$$f^{\#}(w_j^*)(v_i) = w_j^*(f(v_i)) = w_j^*(\sum_{l \in J} m_{li} w_l) = m_{ji}.$$

So $f^{\#}(w_j^*) = \sum_{i \in I} m_{ji} v_i^*$, and hence $f^{\#}$ corresponds to M^{tr}, the transpose of M. (i) and (ii) follow immediately.

(iii) Assume that $\operatorname{Im}(f) \supseteq W$. Let $\alpha \in W^{\#}$ such that $f^{\#}(\alpha) = 0$, i.e. $f^{\#}(\alpha)(v) = 0$, for any $v \in V$. Therefore $\alpha(w_j) = 0$ for any $j \in J$. So $\alpha = 0$, i.e. $f^{\#}$ is injective.

(iv) Assume f is row finite and $\overline{f}$ is injective. It follows by (ii) that $\operatorname{Im}(f^{\#}) \subseteq V^{\#}$. Suppose $\operatorname{Im}(f)$ is a proper subspace of $V^{\#}$. Then there exists a nonzero element $v \in \overline{V} = (V^{\#})^*$ such that $\overline{f^{\#}(\alpha)}(v) = \overline{\overline{\alpha} \circ f}(v) = 0$ for any $\alpha \in W^{\#}$. Therefore by (v) we have $\overline{\alpha}(\overline{f}(v)) = 0$ for any $\alpha \in W^{\#}$, and hence $\overline{f}(v) = 0$, contradicting the injectivity of $\overline{f}$.

(v) We have

$$\overline{g} \circ f(v_i) = \overline{g}(\textstyle\sum_{j \in J} m_{ji} w_j) = \textstyle\sum_{j \in J} m_{ji} g(w_j).$$

Therefore the matrix of $\overline{g} \circ f$ is the product two row finite matrices: the matrices of g and f, and is again row finite. The proof for the equality is straightforward:

$$\begin{aligned} \overline{\overline{g} \circ f}(\textstyle\sum_{i \in I} \lambda_i v_i) &= \textstyle\sum_{i \in I} \lambda_i \overline{g} \circ f(v_i) = \textstyle\sum_{i \in I, j \in J} \lambda_i m_{ji} g(w_j) \\ &= \overline{g}(\textstyle\sum_{i \in I, j \in J} \lambda_i m_{ji} w_j) = \overline{g}(\textstyle\sum_{i \in I} \lambda_i f(v_i)) \\ &= \overline{g} \circ \overline{f}(\textstyle\sum_{i \in I} \lambda_i v_i). \end{aligned}$$

(vi) It follows by (ii) that $\operatorname{Im}(g^{\#}) \subseteq W^{\#}$, and hence $f^{\#} \circ g^{\#}$ is well-defined. For $\beta \in U^{\#}$, we have

$$f^{\#} \circ g^{\#}(\beta) = f^{\#}(\overline{\beta} \circ g) = \overline{\overline{\beta} \circ g} \circ f = \overline{\beta} \circ \overline{g} \circ f = (\overline{g} \circ f)^{\#}(\beta).$$

Hence $(\overline{g} \circ f)^{\#} = \overline{f^{\#}} \circ g^{\#}$. $\square$

A K-linear map $f : V \to W$ can be considered as a formal map from V to W via the embedding $i_W : W \to \overline{W}$.

Corollary 7.3. *Let* $f : V \to W$ *be a row finite* K*-linear map. Then* $f^{\#} : W^{\#} \to V^{\#}$*,* $\alpha \mapsto \alpha \circ f$*, is a row finite* K*-linear map. Moreover, if* $\overline{f}$ *is injective, then* $f^{\#}$ *is surjective; if* f *is surjective, then* $f^{\#}$ *is injective; if* $g : W \to U$ *is a row finite* K*-linear map, then* $(g \circ f)^{\#} = f^{\#} \circ g^{\#}$*.*

Proof. The map $f : V \to W$ is a row finite K-linear map, and hence is a row finite and column finite formal map from V to W. Applying Theorem 7.2 we get the desired results. $\square$

Definition 7.4. *A* formal K-coalgebra *is a K-linear space A together with two formal maps $\Delta : A \to \overline{A \otimes A}$ and $\epsilon : A \to K$ such that*

(i) *the maps Δ and ϵ are row finite,*

(ii) *the following diagrams commute:*

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "\Delta"] & \overline{A \otimes A} \arrow[d, "\overline{\Delta \otimes \mathrm{id}}"] \\ \overline{A \otimes A} \arrow[r, "\overline{\mathrm{id} \otimes \Delta}"] & \overline{A \otimes A \otimes A} \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, hook, "i_A"] & \overline{A \otimes A} \arrow[d, "\overline{\mathrm{id} \otimes \epsilon}"] \\ \overline{A} \arrow[r, "\cong"] & \overline{A \otimes K} \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, hook, "i_A"] & \overline{A \otimes A} \arrow[d, "\overline{\epsilon \otimes \mathrm{id}}"] \\ \overline{A} \arrow[r, "\cong"] & \overline{K \otimes A} \end{tikzcd}$$

We call Δ comultiplication, and ϵ counit.

Remark. (i) A K-coalgebra with row finite comultiplication and row finite counit is a formal K-coalgebra. A formal K-coalgebra with column finite comultiplication is a K-coalgebra.

(ii) Let A be a formal K-coalgebra with comultiplication Δ and counit ϵ. Then $\Delta : A \to \overline{A \otimes A}$ can be lifted to $\overline{\Delta} : \overline{A} \to \overline{A \otimes A}$. The map $\mathrm{id} \otimes \Delta : A \otimes A \to A \otimes \overline{A \otimes A}$ can be lifted to $\overline{A \otimes A \otimes A}$. The same holds for $\Delta \otimes \mathrm{id} : A \otimes A \to \overline{A \otimes A} \otimes A$. $\square$

Definition 7.5. *Let A, B be two formal K-coalgebras with comultiplications Δ_A, Δ_B and counits ϵ_A, ϵ_B respectively. A* formal homomorphism of formal K-coalgebras *from A to B is a formal map $f : A \to \overline{B}$ such that*

(i) *the map f is row finite,*

(ii) *the following diagrams commute:*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\Delta_A"] & \overline{B} \arrow[d, "\overline{\Delta}_B"] \\ \overline{A \otimes A} \arrow[r, "\overline{f \otimes f}"] & \overline{B \otimes B} \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\epsilon_A"] & \overline{B} \arrow[d, "\overline{\epsilon}_B"] \\ K \arrow[r, "\mathrm{id}"] & K \end{tikzcd}$$

If f is column finite, then $f : A \to B$ is called a homomorphism of formal K-coalgebras *from A to B.*

Theorem 7.6. *Let A, B be two formal K-coalgebras with comultiplications Δ_A, Δ_B and counits ϵ_A, ϵ_B respectively.*

(i) *The dual $A^{\#}$ of A with finite support is a K-algebra with multiplication $\Delta_A^{\#}$ and unit $\epsilon_A^{\#}$. Precisely, $\Delta_A^{\#}(f \otimes f')(a) = (f \otimes f')(\Delta_A(a)) = \sum f(a_1)f'(a_2)$, where $f, f' \in A^{\#}$, $a \in A$, and $\Delta_A(a) = \sum a_1 \otimes a_2$ is the Sweedler symbol.*

(ii) *Let $f : A \to \overline{B}$ be a formal homomorphism of formal K-coalgebras from A to B. Then $f^{\#} : B^{\#} \to A^{\#}$ is a homomorphism of K-algebras. Moreover, if $\overline{f}$ is injective, then $f^{\#}$ is surjective; if $\mathrm{Im}(f) \supseteq B$, then $f^{\#}$ is injective.*

(iii) *Let $f : A \to B$ be a homomorphism of formal K-coalgebras. Then $f^{\#} : B^{\#} \to A^{\#}$ is a row finite homomorphism of K-algebras. Moreover, if $\overline{f}$ is injective, then $f^{\#}$ is surjective; if f is surjective, then $f^{\#}$ is injective.*

Proof. (i) Since $\Delta_A : A \to \overline{A \otimes A}$ and $\epsilon_A : A \to K$ are row finite, it follows by Theorem 7.2 (ii) that $\Delta_A^{\#} : (A \otimes A)^{\#} = A^{\#} \otimes A^{\#} \to \overline{A^{\#}}$ and $\epsilon_A^{\#} : K \to \overline{A^{\#}}$ are column finite. Namely, $\Delta_A^{\#} : (A \otimes A)^{\#} = A^{\#} \otimes A^{\#} \to A^{\#}$ and $\epsilon_A^{\#} : K \to A^{\#}$ are two K-linear maps. Now the dual of the diagrams in Definition 7.4 (ii) consists of the axioms for $A^{\#}$ to be an algebra.

(ii) Since f is row finite, it follows by Theorem 7.2 (ii) that $f^{\#} : B^{\#} \to \overline{A^{\#}}$ is column finite, i.e. $f^{\#} : B^{\#} \to A^{\#}$ is a K-linear map. Further, the dual of the diagrams in Definition 7.5 (ii) consists of the axioms for $f^{\#}$ to be an algebra homomorphism. The last two assertions follow from Theorem 7.2 (iii) (iv).

(iii) It follows from (ii) that $f^{\#}$ is an algebra homomorphism, and it follows from Theorem 7.2 (i) that $f^{\#}$ is row finite. The last two assertions follow from Corollary 7.3. □

8 Appendix 2 : Mackey formula for certain endomorphism algebras

This appendix contains a generalization of [8] (4.11).

Let G be a group. Let I be a set with a right G-action such that the stabilizer of any $i \in I$ is a finite subgroup of G.

Let K be a field and V a K-vector space with basis $\{v_i\}_{i\in I}$. Then V is right G-module over K via $v_i^g = v_{ig}$ where $g \in G$, and $i \in I$. Let $B = \mathrm{End}_K(V)$. For i,$j \in I$, let $x_{ij} \in B$ be the K-map defined by $x_{ij}(v_l) = \delta_{jl} v_i$ where $l \in I$. Then

$$B = \{\textstyle\sum_{i,j\in I} \lambda_{ij} x_{ij} | \text{ for fixed } j, \lambda_{ij} = 0 \text{ for almost all } i\}$$

The right G-module structure on V induces a right G-module structure on B satisfying $(bb')^g = b^g (b')^g$ for any $b, b' \in B$ and $g \in G$. Precisely, $(\sum_{i,j\in I} \lambda_{ij} x_{ij})^g = \sum_{i,j\in I} \lambda_{ij} x_{ig,jg}$. Let A be the subring of B with basis $\{x_{ij}\}_{i,j\in I}$. It is easy to see that A is a G-submodule of B.

For a subgroup $H \leq G$, let

$$\begin{aligned} A_H &= \{a \in A | a^h = a, \forall h \in H\} \\ B_H &= \{b \in B | b^h = b, \forall h \in H\}. \end{aligned}$$

For a subgroup chain $H_1 \leq H_2 \leq G$ of G, let

$$T_{H_1,H_2} : A_{H_1} \to B_{H_2}, \quad a \mapsto \sum_{g\in H_1\backslash H_2} a^g.$$

The map T_{H_1,H_2} is well-defined because of the following. Let $a = \sum_{i,j} \lambda_{ij} x_{ij} \in A_{H_1}$, then

$$\begin{aligned} T_{H_1,H_2}(a) &= \textstyle\sum_g (\sum_{i,j} \lambda_{ij} x_{ij})^g = \sum_{i,j} \lambda_{ij} \sum_g (x_{ij})^g \\ &= \textstyle\sum_{i,j,g} \lambda_{ij} x_{ig,jg} = \sum_{p,q} (\sum_{i,j,g: ig=p, jg=q} \lambda_{ij}) x_{pq}. \end{aligned}$$

But for fixed (i, j) and (p, q) there are only finitely many g such that $(ig, jg) = (p, q)$.

Lemma 8.1. *Let $H_1 \leq H_2 \leq G$ be a subgroup chain of G and $a \in A_{H_1}$, $b \in B_{H_2}$.*

(i) *If $ab \in A_{H_1}$, then $T_{H_1,H_2}(ab) = T_{H_1,H_2}(a)b$.*

(ii) *If $ba \in A_{H_1}$, then $T_{H_1,H_2}(ba) = bT_{H_1,H_2}(a)$.*

Proof. We only prove (i); the proof for (ii) is similar. Assume $ab \in A_{H_1}$, then

$$\begin{aligned} T_{H_1,H_2}(ab) &= \sum_{g\in H_1\backslash H_2}(ab)^g = \sum_{g\in H_1\backslash H_2}(a^g b^g) \\ &= \sum_{g\in H_1\backslash H_2}(a^g b) = (\sum_{g\in H_1\backslash H_2} a^g)b \\ &= T_{H_1,H_2}(a)b. \end{aligned}$$

□

Lemma 8.2. *Let $H_1 \leq H_2 \leq H_3 \leq G$ be a subgroup chain of G, and $a \in A_{H_1}$. Assume $T_{H_1,H_2}(a) \in A_{H_2}$, then $T_{H_2,H_3}(T_{H_1,H_2}(a)) = T_{H_1,H_3}(a)$.*

Proof. Let C_1 be a set of representatives of cosets $H_1\backslash H_2$, and C_2 be a set of representatives of cosets $H_2\backslash H_3$, then $C = \{c_1c_2|c_1 \in C_1, c_2 \in C_2\}$ is a set of representatives of cosets $H_1\backslash H_3$. This finishes the proof. □

Lemma 8.3. *Let $H_1 \leq H_3 \leq G$ and $H_2 \leq H_3 \leq G$ be two subgroup chains of G, and $a \in A_{H_1}$. Then*

$$T_{H_1,H_3}(a) = \sum_{w\in H_1\backslash H_3/H_2} T_{H_1^w\cap H_2,H_2}(a^w).$$

Proof. Let W be a set of representatives of $H_1\backslash H_3/H_2$. For each $w \in W$, let C_w be a set of representatives of $(H_1^w \cap H_2)\backslash H_2$. Then $\bigcup_{w\in W} wC_w$ is a set of representatives of $H_1\backslash H_3$. Indeed, for $h \in H_3$ there exists $h_1 \in H_1$, $w \in W$ and $h_2 \in H_2$ such that $h = h_1wh_2$. Since $h_2 \in H_2$, there exists $g \in H_1^w \cap H_2$ and $c \in C_w$ such that $h_2 = gc$. Thus $h = h_1wgc = h_1wgw^{-1}wc \in H_1wc$. Conversely, suppose $wc = w'c'$ where $w, w' \in W$, $c \in C_w$ and $c' \in C_{w'}$. Then $w = w'c'c^{-1}$. Since W is a set of representatives of $H_1\backslash H_3/H_2$, and $c'c^{-1} \in H_2$, we have $w = w'$ and $c = c'$. □

Lemma 8.4. *(Mackey's formula) Let $H_1 \leq H_3 \leq G$ and $H_2 \leq H_3 \leq G$ be two subgroups chains of G, and $a \in A_{H_1}$, $b \in A_{H_2}$. If $aT_{H_2,H_3}(b) \in A_{H_1}$, $T_{H_1\cap H_2^w,H_1}(ab^w) = 0$ for almost all w, and $T_{H_1\cap H_2^w,H_1}(ab^w) \in A_{H_1}$ for all w, then*

$$T_{H_1,H_3}(a)T_{H_2,H_3}(b) = \sum_{w\in H_1\backslash H_3/H_2} T_{H_1\cap H_2^w,H_3}(ab^w).$$

Proof. First note that $b \in A_{H_2}$ implies $b^w \in A_{H_2^w}$. Thus a and b^w are in $A_{H_1\cap H_2^w}$ and so is their product ab^w.

By Lemma 8.3 and Lemma 8.1 we have

$$\begin{aligned} aT_{H_2,H_3}(b) &= a\sum_{w\in H_2\backslash H_3/H_1} T_{H_1\cap H_2^w,H_1}(b^w) \\ &= \sum_{w\in H_2\backslash H_3/H_1} T_{H_1\cap H_2^w,H_1}(ab^w). \end{aligned}$$

Since $aT_{H_2,H_3}(b) \in A_{H_1}$, it follows that

$$\begin{aligned} T_{H_1,H_3}(a)T_{H_2,H_3}(b) &= T_{H_1,H_3}(aT_{H_2,H_3}(b)) \\ &= T_{H_1,H_3}(\sum_{w\in H_2\backslash H_3/H_1} T_{H_1\cap H_2^w,H_1}(ab^w)) \\ &= \sum_{w\in H_2\backslash H_3/H_1} T_{H_1,H_3}(T_{H_1\cap H_2^w,H_1}(ab^w)) \\ &= \sum_{w\in H_2\backslash H_3/H_1} T_{H_1\cap H_2^w,H_3}(ab^w), \end{aligned}$$

where the last equality follows by Lemma 8.2. □

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